Smoothed Particle Hydrodynamics (SPH) An Introduction

Simulation in Computer Graphics
University of Freiburg

WS 08/09

not relevant for the oral exam

Acknowledgement

This slide set is based on:

- J. Monaghan, "Smoothed Particle Hydrodynamics", *Reports on Progress in Physics* 68, 8, pp. 1703-1759, 2005.
- M. Müller, D. Charypar, M. Gross, "Particle-Based Fluid Simulation for Interactive Applications", *Proc. ACM SIGGRAPH / Eurographics SCA*, 2003.
- M. Becker, M. Teschner, "Weakly Compressible SPH for Free Surface Flows", *Proc. ACM SIGGRAPH / Eurographics SCA* 2007.
- R. Bridson, M. Müller-Fischer, "Fluid Simulation", *ACM SIGGRAPH 2007*, course notes.
- Franz J. Vesely, *"Computational Physics – An Introduction"*, Kluwer, New York, ISBN 0-306-46631-7.

Motivation

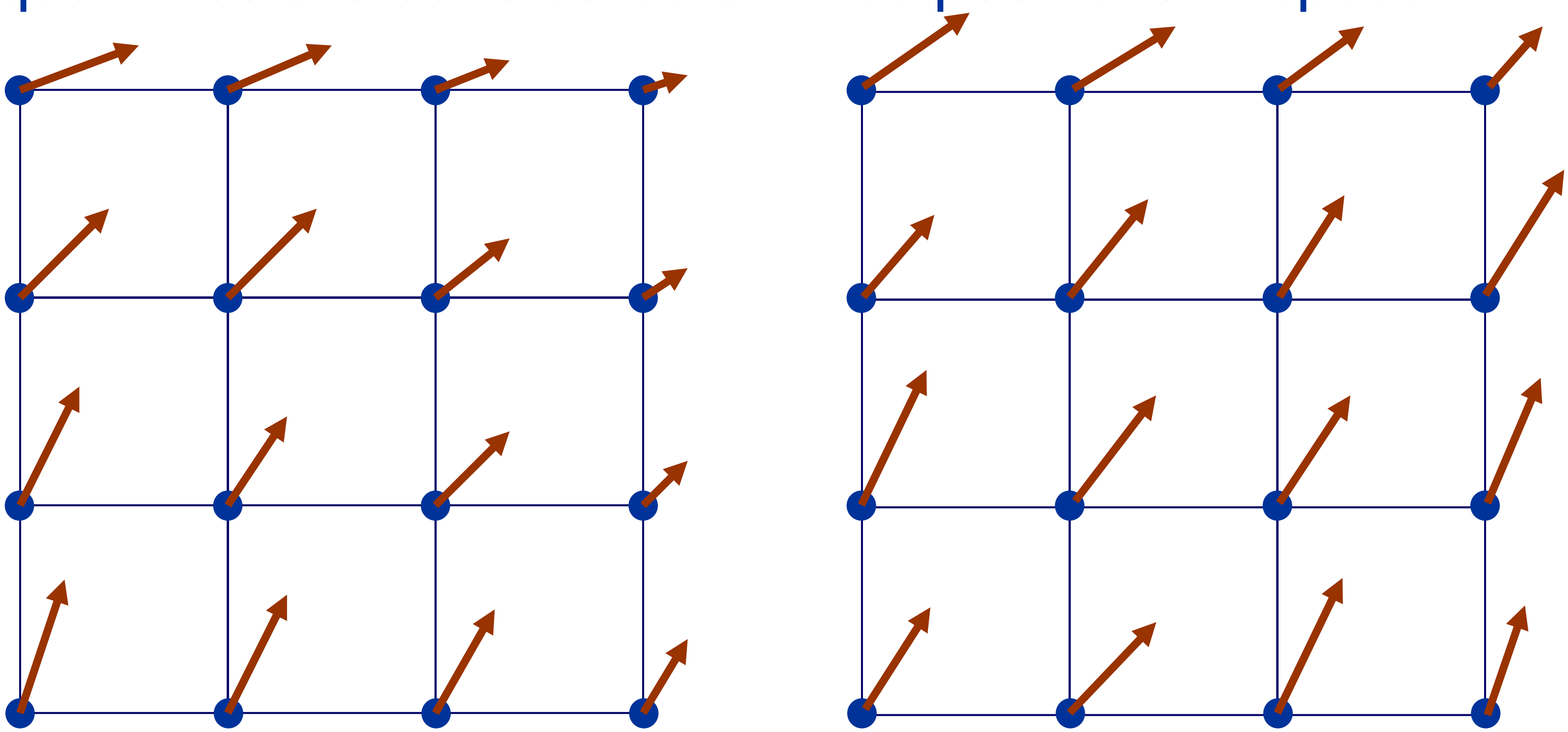
M. Becker, M. Teschner, Weakly Compressible SPH for Free Surface Flow,
Proc. ACM SIGGRAPH / Eurographics SCA, San Diego, CA, USA, pp. 63-72, 2007.

Outline

- Eulerian and Lagrangian viewpoints
- governing fluid equations for SPH
- spatial discretization in SPH
- spatial derivatives in SPH
- parameters
- SPH variants and additional fluid properties
- implementation and results

Eulerian Approach

- quantities are considered at fixed positions in space



v (x,y,z,t)

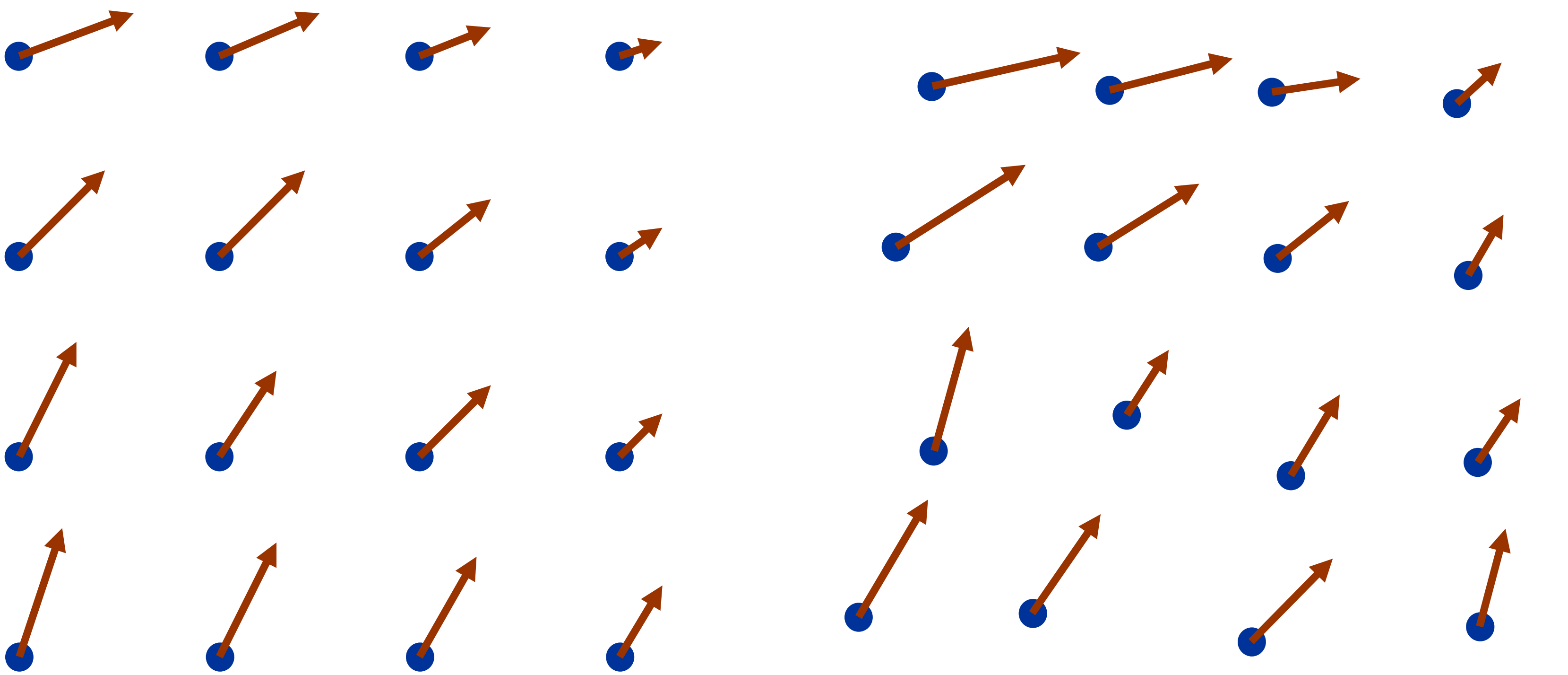
$\rho\,(x,y,z,t)$

v $(x,y,z,t+\Delta t)$

$\rho\,(x,y,z,t+\Delta t)$

Lagrangian Approach

- quantities move with the flow (particle system)
- SPH is Lagrangian



$\mathbf{v}\,(x,y,z,t\,)$

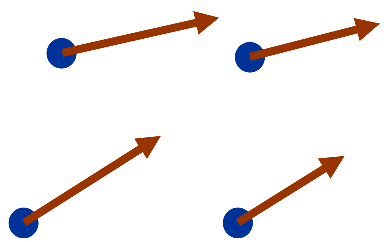
$\rho\,(x,y,z,t\,)$

$\mathbf{v}\,(x+\Delta t\cdot u,\ y+\Delta t\cdot v,\ z+\Delta t\cdot w,\ t+\Delta t\,)$

$\rho\,(x+\Delta t\cdot u,\ y+\Delta t\cdot v,\ z+\Delta t\cdot w,\ t+\Delta t\,)$

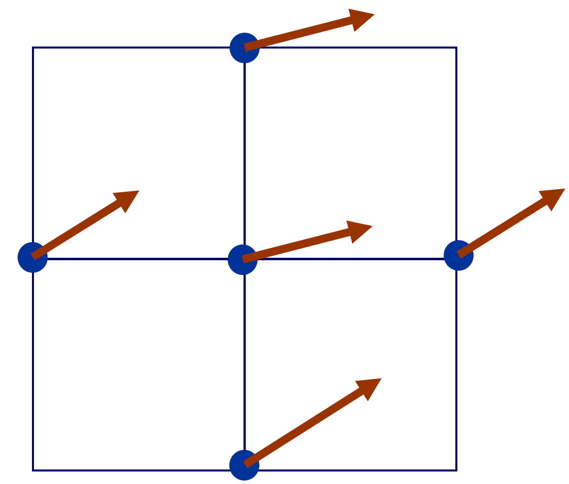
Substantial Derivative

- Lagrangian and Eulerian viewpoint



time rate of change of a quantity A of a moving fluid element

$$\frac{DA}{Dt} \equiv \frac{\partial A}{\partial t} + A(\mathbf{v} \cdot \nabla)$$



time rate of change of a quantity A at a fixed position

- in Eulerian approaches, the right-hand side is used to map everything onto a spatial grid
- in Lagrangian approaches, the left-hand side is used (position, velocity and density of a particle is traced)

Outline

- Eulerian and Lagrangian viewpoints
- governing fluid equations for SPH
- spatial discretization in SPH
- spatial derivatives in SPH
- parameters
- SPH variants and additional fluid properties
- implementation and results

Governing Equations

- governing equations in non-conservation form
  consider small moving fluid elements
- time rate of change of the density (continuity equation)

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{v}$$

- time rate of change of the velocity (momentum equation)

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho} \nabla \mathbf{p}$$

- time rate of change of the position

$$\frac{D\mathbf{x}}{Dt} = \mathbf{v}$$

Problem

- how to compute the spatial derivatives $\nabla \cdot \mathbf{v}$ $\nabla \mathbf{p}$
- if we had a grid, this would be easy
- SPH provides a concept to approximate the spatial derivatives using particles

Outline

- Eulerian and Lagrangian viewpoints
- governing fluid equations for SPH
- spatial discretization in SPH
- spatial derivatives in SPH
- parameters
- SPH variants and additional fluid properties
- implementation and results

Spatial Discretization

- continuous quantity A(**x**) can be written as convolution of the quantity and a delta function

$$A(\mathbf{x}) = \int A(\mathbf{x}') \cdot \delta(\mathbf{x} - \mathbf{x}')d\mathbf{x}'$$

- the delta function is approximated with a kernel function *W* of limited support *h*

$$A(\mathbf{x}) \approx \int A(\mathbf{x}') \cdot W(\mathbf{x} - \mathbf{x}', h)d\mathbf{x}'$$

- *W* should be differentiable, normalized, and converge to a delta function

$$\int W(\mathbf{x} - \mathbf{x}', h)d\mathbf{x}' = 1 \qquad \lim_{h \to 0} W(\mathbf{x} - \mathbf{x}', h) = \delta(\mathbf{x} - \mathbf{x}')$$

Spatial Discretization

- multiply the approximation with ρ / ρ

$$A(\mathbf{x}) \approx \int \frac{A(\mathbf{x}')}{\rho(\mathbf{x}')} \cdot W(\mathbf{x}-\mathbf{x}', h)\,\rho(\mathbf{x}')\,d\mathbf{x}'$$

- consider a limited number of particles

$$A_a = \sum_b m_b \frac{A_b}{\rho_b} \cdot W(\mathbf{x_a}-\mathbf{x_b}, h)$$

- quantity A_a at an arbitrary position $\mathbf{x}_a$ is approximated using quantities A_b at sample positions $\mathbf{x}_b$
- only particles with $|\mathbf{x_a}-\mathbf{x_b}| < h$ are considered
- the kernel function *W* realizes a diminishing influence of particles at a larger distance

Outline

- Eulerian and Lagrangian viewpoints
- governing fluid equations for SPH
- spatial discretization in SPH
- spatial derivatives in SPH
- parameters
- SPH variants and additional fluid properties
- implementation and results

Spatial Derivatives

- if m_b, ρ_b, A_b are considered constant with respect to space, then

$$\nabla A_a = \sum_b m_b \frac{A_b}{\rho_b} \cdot \nabla W(\mathbf{x_a} - \mathbf{x_b}, h) = \sum_b m_b \frac{A_b}{\rho_b} \cdot \nabla W_{ab}$$

- alternatively,

$$\nabla A_a = \frac{1}{\rho_a} \sum_b m_b (A_b - A_a) \cdot \nabla W_{ab}$$

is symmetric and less sensitive to particle disorder

Spatial Derivatives

- now, the continuity equation can be written as

$$\frac{d\rho_a}{dt} = -\rho_a \sum_b \frac{m_b}{\rho_b} \mathbf{v}_b \cdot \nabla W_{ab}$$

- the alternative, symmetric form would be

$$\frac{d\rho_a}{dt} = -\rho_a \frac{1}{\rho_a} \sum_b m_b \mathbf{v}_{ba} \cdot \nabla W_{ab} = \sum_b m_b \mathbf{v}_{ab} \cdot \nabla W_{ab}$$

Spatial Derivatives

- the momentum equation results in

$$\frac{d\mathbf{v_a}}{dt} = -\frac{1}{\rho_a}\sum_b m_b \frac{p_b}{\rho_b}\nabla W_{ab}$$

- symmetrization using

$$\frac{d\mathbf{v_a}}{dt} = -\frac{1}{\rho_a}\nabla p_a = -\nabla\left(\frac{p_a}{\rho_a}\right) + \frac{p_a}{\rho_a^2}\nabla\rho$$

results in

$$\frac{d\mathbf{v_a}}{dt} = -\sum_b m_b \left(\frac{p_b}{\rho_b^2} + \frac{p_a}{\rho_a^2}\right)\nabla W_{ab}$$

Outline

- Eulerian and Lagrangian viewpoints
- governing fluid equations for SPH
- spatial discretization in SPH
- spatial derivatives in SPH
- parameters
- SPH variants and additional fluid properties
- implementation and results

Kernel Function

- cubic spline

$$W(\mathbf{x}_{\mathbf{ab}}, h) = \frac{\sigma}{h^d} \begin{cases} 6q^3 - 6q^2 + 1 & 0 \le q \le \frac{1}{2} \\ 2(1-q)^3 & \frac{1}{2} \le q \le 1 \\ 0 & q > 1 \end{cases} \qquad q = |\mathbf{x}_{\mathbf{ab}}| / h$$

with *d* denoting the dimensionality of the simulation and

$$\sigma = \begin{cases} \frac{4}{3} & d = 1 \\ \frac{40}{7\pi} & d = 2 \\ \frac{8}{\pi} & d = 3 \end{cases}$$

Kernel Function

- its derivative

$$\frac{\partial W(\mathbf{x}_{ab}, h)}{\partial \mathbf{x}_{ab}} = \frac{6\sigma}{h^{d+1}} \begin{cases} 3q^2 - 2q & 0 \le q \le \frac{1}{2} \\ -(1-q)^2 & \frac{1}{2} \le q \le 1 \\ 0 & q > 1 \end{cases} \qquad q = |\mathbf{x}_{ab}| / h$$

resulting in

$$\nabla W(\mathbf{x}_{ab}, h) = \frac{\partial W(\mathbf{x}_{ab}, h)}{\partial \mathbf{x}_{ab}} \frac{\mathbf{x}_{ab}}{|\mathbf{x}_{ab}|}$$

Smoothing Length

- the smoothing length h is commonly chosen twice the initial distance of adjacent particles

State Equation

- in order to update the quantities of the particles, we need to know the pressure *p*

$$\frac{d\mathbf{x}_\mathbf{a}}{dt} = \mathbf{v}_\mathbf{a} \qquad \frac{d\mathbf{v}_\mathbf{a}}{dt} = -\sum_b m_b \left(\frac{p_b}{\rho_b^2} + \frac{p_a}{\rho_a^2} \right) \nabla W_{ab} \qquad \frac{d\rho_a}{dt} = \sum_b m_b \mathbf{v}_{ab} \cdot \nabla W_{ab}$$

- various state equations relate pressure *p* with density ρ
  - ideal gas equation (compressible)

    $$p = k\rho$$

    $$p = k(\rho - \rho_0)$$

    with ρ_0 being the initial density of the fluid
  - Tait equation (weakly compressible)

    $$p = k_1 \left(\left(\frac{\rho}{\rho_0} \right)^{k_2} - 1 \right)$$

Outline

- Eulerian and Lagrangian viewpoints
- governing fluid equations for SPH
- spatial discretization in SPH
- spatial derivatives in SPH
- parameters
- SPH variants and additional fluid properties
- implementation and results

Density Update

- in contrast to the differential density update,

$$\frac{d\rho_a}{dt} = \sum_b m_b \mathbf{v}_{ab} \cdot \nabla W_{ab}$$

density can be computed explicitly

$$\rho_a = \sum_b m_b \frac{\rho_b}{\rho_b} \cdot W_{ab} = \sum_b m_b W_{ab}$$

Artificial Viscosity

- if the Euler equation is discretized, artificial viscosity can be added in various forms, e. g.

$$\frac{d\mathbf{v}_{\mathbf{a}}}{dt} = -\sum_b m_b \Pi_{ab} \nabla W_{ab}$$

$$\Pi_{ab} = \begin{cases} \dfrac{-\alpha 2 c_s \mu_{ab} + \beta \mu_{ab}^2}{\rho_a + \rho_b} & \mathbf{v}_{\mathbf{ab}} \mathbf{x}_{\mathbf{ab}} < 0 \\ 0 & \mathbf{v}_{\mathbf{ab}} \mathbf{x}_{\mathbf{ab}} \geq 0 \end{cases}$$

$$\mu_{ab} = \frac{h \cdot \mathbf{v}_{\mathbf{ab}} \cdot \mathbf{x}_{\mathbf{ab}}}{\mathbf{x}_{\mathbf{ab}}^2 + \varepsilon h^2} \qquad c^2 = \gamma \rho_0 \rho^{\gamma - 1}$$

α, β, γ are user-defined

- basically damps relative velocities of particles moving towards each other

Artificial Viscosity

- there exist simplified alternatives
- 1.

$$\frac{d\mathbf{v}_{\mathbf{a}}}{dt} = -\sum_b m_b \Pi_{ab} \nabla W_{ab}$$

$$\Pi_{ab} = \begin{cases} -\dfrac{4\alpha h c_s}{\rho_a + \rho_b} \cdot \dfrac{\mathbf{v}_{\mathbf{ab}} \mathbf{x}_{\mathbf{ab}}}{\left|\mathbf{x}_{\mathbf{ab}}\right|^2 + \varepsilon h^2} & \mathbf{v}_{\mathbf{ab}} \mathbf{x}_{\mathbf{ab}} < 0 \\ 0 & \mathbf{v}_{\mathbf{ab}} \mathbf{x}_{\mathbf{ab}} \geq 0 \end{cases}$$

- 2.

$$\frac{d\mathbf{v}_{\mathbf{a}}}{dt} = -\frac{\mu}{m_a} \sum_b m_b \frac{\mathbf{v}_{\mathbf{ab}}}{\rho_b} \nabla^2 W_{ab}$$

Surface Tension

- surface tension forces are intended to minimize the surface curvature
- models commonly employ so-called color values c_a

$$c_a = \sum_b \frac{c_b m_b}{\rho_b} W_{ab} \qquad \frac{d\mathbf{v}_\mathbf{a}}{dt} = -\kappa \nabla^2 c_a \frac{\nabla c_a}{|\nabla c_a|}$$

- κ is user-defined, ∇c_a approximates the surface normal
- alternatively, additional attraction forces could be introduced (cohesion forces)

$$\frac{d\mathbf{v}_\mathbf{a}}{dt} = -\frac{\kappa}{m_a} \sum_b m_b W_{ab} (\mathbf{x}_\mathbf{a} - \mathbf{x}_\mathbf{b})$$

κ is user-defined

Outline

- Eulerian and Lagrangian viewpoints
- governing fluid equations for SPH
- spatial discretization in SPH
- spatial derivatives in SPH
- parameters
- SPH variants and additional fluid properties
- implementation and results

Simulation Step

- define kernel function *W*
- initialize positions **x**, velocities **v**, masses *m*, smoothing length *h*

- density update $\rho_a = \sum_b m_b W_{ab}$
- pressure computation (gas equation, tait equation)
- velocity update $\frac{d\mathbf{v_a}}{dt} = -\sum_b m_b \left(\frac{p_b}{\rho_b^2} + \frac{p_a}{\rho_a^2} + \Pi_{ab} \right) \nabla W_{ab}$
- position update $\frac{d\mathbf{x_a}}{dt} = \mathbf{v_a}$

Implementation Issues

- performance is dominated by the efficient search for pairs of interacting particles
- spatial data structures are commonly employed, e. g. uniform spatial grids
- visualization
  - explicit particle rendering
  - iso-surface rendering
    (Marching Cubes algorithm for surface reconstruction)

Results

M. Becker, M. Teschner, Weakly Compressible SPH for Free Surface Flow,
Proc. ACM SIGGRAPH / Eurographics SCA, San Diego, CA, USA, pp. 63-72, 2007.

Results

M. Becker, M. Teschner, Weakly Compressible SPH for Free Surface Flow,
Proc. ACM SIGGRAPH / Eurographics SCA, San Diego, CA, USA, pp. 63-72, 2007.