Advanced Computer Graphics

Ray-Object Intersections

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Motivation

- rays
  - a half-line specified by an origin/position $\mathbf{o}$ and a direction $\mathbf{d}$
  - parametric form $\mathbf{r}(t) = \mathbf{o} + t\mathbf{d}$ with $0 \leq t \leq \infty$
- nearest intersection with all objects has to be computed, i.e., intersection with minimal $t \geq 0$
- in implementations, usually $t \geq \varepsilon$ to avoid self-intersections, e.g., if rays start at object surfaces
Outline

- implicit surfaces
- parametric surfaces
- combined objects
- triangles
- axis-aligned boxes
- isosurfaces in grids
Implicit Surfaces

- implicit functions implicitly define a set of surface points
- for a surface point \((x, y, z)\), an implicit function \(f(x, y, z)\) is zero
- an intersection occurs, if a point on a ray satisfies the implicit equation \(f(x, y, z) = f(r(t)) = f(o + td) = 0\)
- e.g., all points \(p\) on a plane with surface normal \(n\) and offset \(r\) satisfy the equation \(n \cdot (p - r) = 0\)
- the intersection with a ray can be computed based on \(t\)
  \[
  n \cdot (o + td - r) = 0
  \]
  \[
  t = \frac{(r - o) \cdot n}{n \cdot d}
  \]
Implicit Surfaces - Normal

- perpendicular to the surface
- given by the gradient of the implicit function

\[ \mathbf{n} = \nabla f(\mathbf{p}) = \left( \frac{\partial f(\mathbf{p})}{\partial x}, \frac{\partial f(\mathbf{p})}{\partial y}, \frac{\partial f(\mathbf{p})}{\partial z} \right) \]

- e.g., for a point \( \mathbf{p} \) on a plane \( f(\mathbf{p}) = \mathbf{n} \cdot (\mathbf{p} - \mathbf{r}) = 0 \)

\[ \mathbf{n} = \nabla f(\mathbf{p}) = (n_x, n_y, n_z) \]
Quadrics

- e.g.
  - sphere \( \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{a^2} - 1 = 0 \)
  - ellipsoid \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \)
  - paraboloid \( \frac{x^2}{a^2} + \frac{y^2}{a^2} - z = 0 \)
  - hyperboloid \( \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0 \)
  - cone \( \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \)
  - cylinder \( \frac{x^2}{a^2} + \frac{y^2}{a^2} - 1 = 0 \)

- represented by quadratic equations, i.e. zero, one or two intersections with a ray

[Wikipedia: Quadric]
Quadrics - Sphere

- at the origin with radius one \( f(\mathbf{p}) = x^2 + y^2 + z^2 - 1 = 0 \)
  \[(o_x + td_x)^2 + (o_y + td_y)^2 + (o_z + td_z)^2 - 1 = 0\]
- quadratic equation in \( t \)
  \[A t^2 + B t + C = 0 \quad A = d_x^2 + d_y^2 + d_z^2\]
  \[B = 2(d_x o_x + d_y o_y + d_z o_z) \quad C = o_x^2 + o_y^2 + o_z^2 - 1\]
  \[t_{0,1} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}\]
- surface normal
  \[\mathbf{n} = \nabla f(\mathbf{p}) = (2x, 2y, 2z)\]
- the sphere can be combined with arbitrary affine transformations
Quadrics - Example
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Parametric Surfaces

- are represented by functions with 2D parameters
  \[ x = f(u, v) \quad y = g(u, v) \quad z = h(u, v) \]

- intersection is computed using a linear system with three equations and three unknowns \( t, u, v \)
  \[ o_x + td_x = f(u, v) \quad o_y + td_y = g(u, v) \quad o_z + td_z = h(u, v) \]

- normal vector
  \[ \mathbf{n}(u, v) = \left( \frac{\partial f}{\partial u}, \frac{\partial g}{\partial u}, \frac{\partial h}{\partial u} \right) \times \left( \frac{\partial f}{\partial v}, \frac{\partial g}{\partial v}, \frac{\partial h}{\partial v} \right) \]
Parametric Surfaces, e.g., Cylinder, Sphere

- cylinder about $z$-axis with parameters $\phi$ and $\nu$
  \[
  x = \cos \phi \quad 0 \leq \phi \leq 2\pi \\
  y = \sin \phi \\
  z = z_{\text{min}} + \nu(z_{\text{max}} - z_{\text{min}}) \quad 0 \leq \nu \leq 1
  \]

- sphere centered at the origin with parameters $\phi$ and $\theta$
  \[
  x = \cos \phi \sin \theta \quad 0 \leq \phi \leq 2\pi \\
  y = \sin \phi \sin \theta \quad 0 < \theta \leq \pi \\
  z = \cos \theta
  \]

- parametric representations are used to render partial objects, e.g. $\phi_{\text{min}} \leq \phi \leq \phi_{\text{max}}$

- can be combined with arbitrary affine transformations
Parametric Surfaces, e.g., Disk, Cone

- Disk with radius $r$ at height $h$ along the $z$-axis with inner radius $r_i$ with parameters $u$ and $\nu$

  $$\phi = u\phi_{\text{max}} \quad 0 \leq u \leq 1$$
  $$x = ((1 - \nu)r_i + \nu r) \cos \phi \quad 0 \leq \nu \leq 1$$
  $$y = ((1 - \nu)r_i + \nu r) \sin \phi$$
  $$z = h$$

- Cone with radius $r$ and height $h$ and parameters $u$ and $\nu$

  $$\phi = u\phi_{\text{max}} \quad 0 \leq u \leq 1$$
  $$x = r(1 - \nu) \cos \phi \quad 0 \leq \nu \leq 1$$
  $$y = r(1 - \nu) \sin \phi$$
  $$z = \nu h$$
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Compound Objects

- consist of components
**Constructive Solid Geometry**

- combine simple objects to complex geometry using Boolean operators

[Wikipedia: Constructive Solid Geometry]
[Wikipedia: Computergrafik]

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Triangle

- parametric representation (based on barycentric coords)
  \[ \mathbf{p}(b_1, b_2) = (1 - b_1 - b_2)\mathbf{p}_0 + b_1\mathbf{p}_1 + b_2\mathbf{p}_2 \]
  \[ b_1 \geq 0 \quad b_2 \geq 0 \quad b_1 + b_2 \leq 1 \]

- intersection is computed using a linear system
  \[ \mathbf{o} + t\mathbf{d} = (1 - b_1 - b_2)\mathbf{p}_0 + b_1\mathbf{p}_1 + b_2\mathbf{p}_2 \]

- solution (for non-degenerated triangles not parallel to the ray)

\[
\begin{pmatrix}
  t \\
  b_1 \\
  b_2 \\
\end{pmatrix}
= \frac{1}{(\mathbf{d} \times \mathbf{e}_2) \cdot \mathbf{e}_1}
\begin{pmatrix}
  (\mathbf{s} \times \mathbf{e}_1) \cdot \mathbf{e}_2 \\
  (\mathbf{d} \times \mathbf{e}_2) \cdot \mathbf{s} \\
  (\mathbf{s} \times \mathbf{e}_1) \cdot \mathbf{d} \\
\end{pmatrix}
\]

\[ \mathbf{e}_1 = \mathbf{p}_1 - \mathbf{p}_0 \]
\[ \mathbf{e}_2 = \mathbf{p}_2 - \mathbf{p}_0 \]
\[ \mathbf{s} = \mathbf{o} - \mathbf{p}_0 \]
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Axis-Aligned (Bounding) Box AABB

- are commonly used to enclose complex geometry
- accelerate the ray-object intersection
  - if a ray misses the simple box, the enclosed complex object cannot be hit by the ray
- AABBs are aligned with the principal axes of a coordinate system
  - to simplify intersection tests
  - to simplify the representation (two points represent three slabs in x-, y-, z-direction)
Axis-Aligned (Bounding) Box AABB

- boxes are defined by slabs
- intersections of rays with slabs are analyzed to check for ray-box intersection
  - e.g. non-overlapping ray intervals within different slabs indicate that the ray misses the box

\[
\mathbf{n} \cdot (\mathbf{o} + td - \mathbf{r}) = 0
\]
\[
t = \frac{(\mathbf{r} - \mathbf{o}) \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{d}}
\]
\[
(1, 0, 0)^T \cdot (\mathbf{o} + td - (x_{0,1}, 0, 0)^T) = 0
\]
\[
t \cdot x_{\min, x_{\max}} = \frac{(x_{0,1} - o_x)}{d_x}
\]
Axis-Aligned (Bounding) Box AABB

- Overlapping ray intervals indicate intersections, e.g. $t_{x_{\min}} < t_{y_{\max}} \land t_{x_{\max}} > t_{y_{\min}} \Rightarrow$ intersection (largest entering value $t$ is smaller than the smallest leaving value $t$, only positive values $t$ are considered).
Axis-Aligned (Bounding) Box AABB

- negative entering values and positive leaving values for all axes indicate that the ray starts inside the box

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[Suffern]
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Setting

general setting

setting with a uniform grid

ray equation:
\[ x = x_a + t x_b \]
\[ y = y_a + t y_b \]
\[ z = z_a + t z_b \]

\[ \rho(x, y, z) = \rho_{iso} \]
Scalar-field Interpolation

- trilinear interpolation of scalar values inside a grid cell

\[
\rho(u, v, w) = (1 - u)(1 - v)(1 - w)\rho_{000} +
(1 - u)(1 - v)w\rho_{001} +
(1 - u)v(1 - w)\rho_{010} +
(1 - u)v(1 - w)\rho_{100} +
u(1 - v)(1 - w)\rho_{101} +
u(1 - v)w\rho_{110} +
u v(1 - w)\rho_{111} +
u v(1 - w)\rho_{111} +
\]

\[
u = \frac{x - x_0}{x_1 - x_0}, \quad v = \frac{y - y_0}{y_1 - y_0}, \quad w = \frac{z - z_0}{z_1 - z_0}
\]

[Parker et al.]
Isosurface Normal

- gradient of the scalar field

\[ \mathbf{n} = \nabla \rho(x, y, z) = \left( \frac{\partial \rho(x, y, z)}{\partial x}, \frac{\partial \rho(x, y, z)}{\partial y}, \frac{\partial \rho(x, y, z)}{\partial z} \right)^T \]

- approximated, e.g., with finite difference

\[ n_x = \sum_{i,j,k=0,1} \frac{(-1)^{i+1} v_j w_k}{x_1 - x_0} \rho_{ijk} \]

\[ n_y = \sum_{i,j,k=0,1} \frac{(-1)^{j+1} u_i w_k}{y_1 - y_0} \rho_{ijk} \]

\[ n_z = \sum_{i,j,k=0,1} \frac{(-1)^{k+1} u_i v_j}{z_1 - z_0} \rho_{ijk} \]
Ray-Isosurface Intersection

- ray \( r(t) = o + td \)
- isosurface intersection
  - compute \( t \) with \( \rho(r(t)) = \rho_{iso} = \rho(o + td) \)
- cubic polynomial in \( t \)
Application

- rendering of surfaces in particle-based fluid simulations with large particle counts
Summary

if ray-object intersections and the respective surface normal can be computed, the object can be rendered in a ray tracer

- implicit surfaces
- parametric surfaces
  - used for partial objects
- combinations of objects
- triangles
- axis-aligned boxes
  - for accelerated ray-object intersections in case of complex geometry