Advanced Computer Graphics
Ray-Object Intersections

Matthias Teschner

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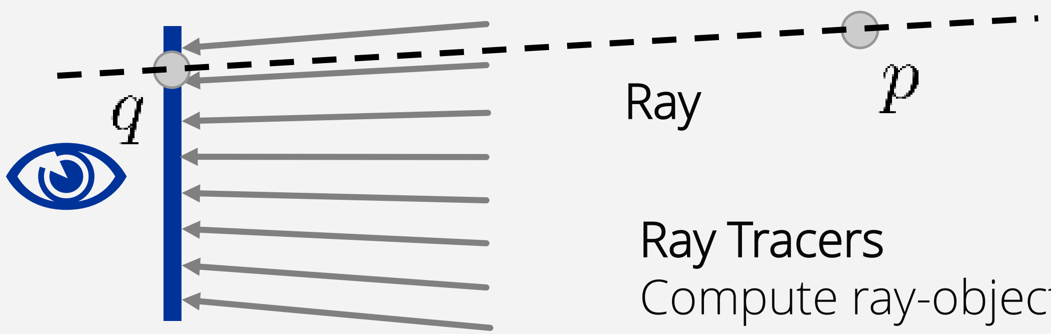
Rendering

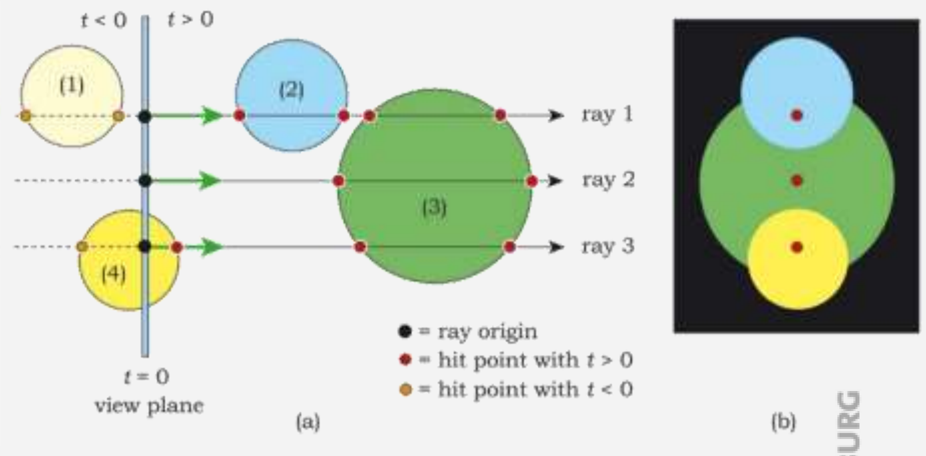
- What is visible in an image?
  - Transformations from model space to screen space
  - Ray-object intersections
- Which color is it?
  - Shading/Lighting (e.g. Phong model)
  - Rendering equation



Ray Tracers
Compute ray-object intersections to estimate q from p

Motivation

- Rays
  - A half-line specified by an origin / position **o** and a direction **d**
  - Parametric form $\mathbf{r}(t) = \mathbf{o} + t\mathbf{d}$ with $0 \leq t \leq \infty$
- Nearest intersection with all objects has to be computed, i.e. intersection with minimal $t \geq 0$
- In implementations, usually $t \geq \varepsilon$ to avoid self-intersections, e.g., if rays start at object surfaces



[Suffern]

Outline

- Implicit surfaces
- Parametric surfaces
- Combined objects
- Triangles
- Axis-aligned boxes
- Isosurfaces in grids

Implicit Surfaces

- Implicit functions implicitly define a set of surface points
- For a surface point (x,y,z), an implicit function *f*(x,y,z) is zero
- An intersection occurs, if a point on a ray satisfies the implicit equation $f(x,y,z) = f(\mathbf{r}(t)) = f(\mathbf{o} + t\mathbf{d}) = 0$
- E.g., all points **p** on a plane with surface normal **n** and offset **r** satisfy the equation $\mathbf{n} \cdot (\mathbf{p} - \mathbf{r}) = 0$
- The intersection with a ray can be computed based on *t*

$\mathbf{n} \cdot (\mathbf{o} + t\mathbf{d} - \mathbf{r}) = 0 \quad t = \frac{(\mathbf{r}-\mathbf{o})\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{d}}$ if **d** is not orthogonal to **n**

Implicit Surfaces - Normal

– Perpendicular to the surface
– Given by the gradient of the implicit function

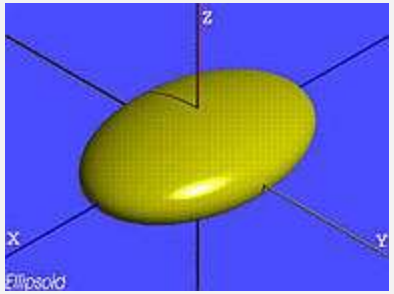
$$\mathbf{n} = \nabla f(\mathbf{p}) = \left(\frac{\partial f(\mathbf{p})}{\partial x}, \frac{\partial f(\mathbf{p})}{\partial y}, \frac{\partial f(\mathbf{p})}{\partial z} \right)$$

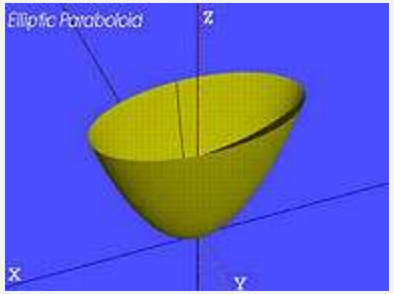
– E.g., for a point p on a plane $f(\mathbf{p}) = \mathbf{n} \cdot (\mathbf{p} - \mathbf{r}) = 0$

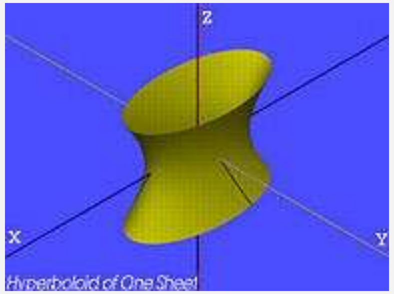
$$\mathbf{n} = \nabla f(\mathbf{p}) = (n_x, n_y, n_z)$$

Quadrics

- E.g.
  - Sphere $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{a^2} - 1 = 0$
  - Ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$
  - Paraboloid $\frac{x^2}{a^2} + \frac{y^2}{a^2} - z = 0$
  - Hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0$
  - Cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$
  - Cylinder $\frac{x^2}{a^2} + \frac{y^2}{a^2} - 1 = 0$
- Represented by quadratic equations, i.e. zero, one or two intersections with a ray







[Wikipedia: Quadric]

Quadrics - Sphere

- At the origin with radius one $f(\mathbf{p}) = x^2 + y^2 + z^2 - 1 = 0$

  $$(o_x + td_x)^2 + (o_y + td_y)^2 + (o_z + td_z)^2 - 1 = 0$$

- Quadratic equation in *t*

  $$At^2 + Bt + C = 0 \qquad A = d_x^2 + d_y^2 + d_z^2 \quad B = 2(d_x o_x + d_y o_y + d_z o_z)$$

  $$t_{0,1} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \qquad C = o_x^2 + o_y^2 + o_z^2 - 1$$

- Surface normal

  $$\mathbf{n} = \nabla f(\mathbf{p}) = (2x, 2y, 2z)$$

- The sphere can be combined with arbitrary affine transformations

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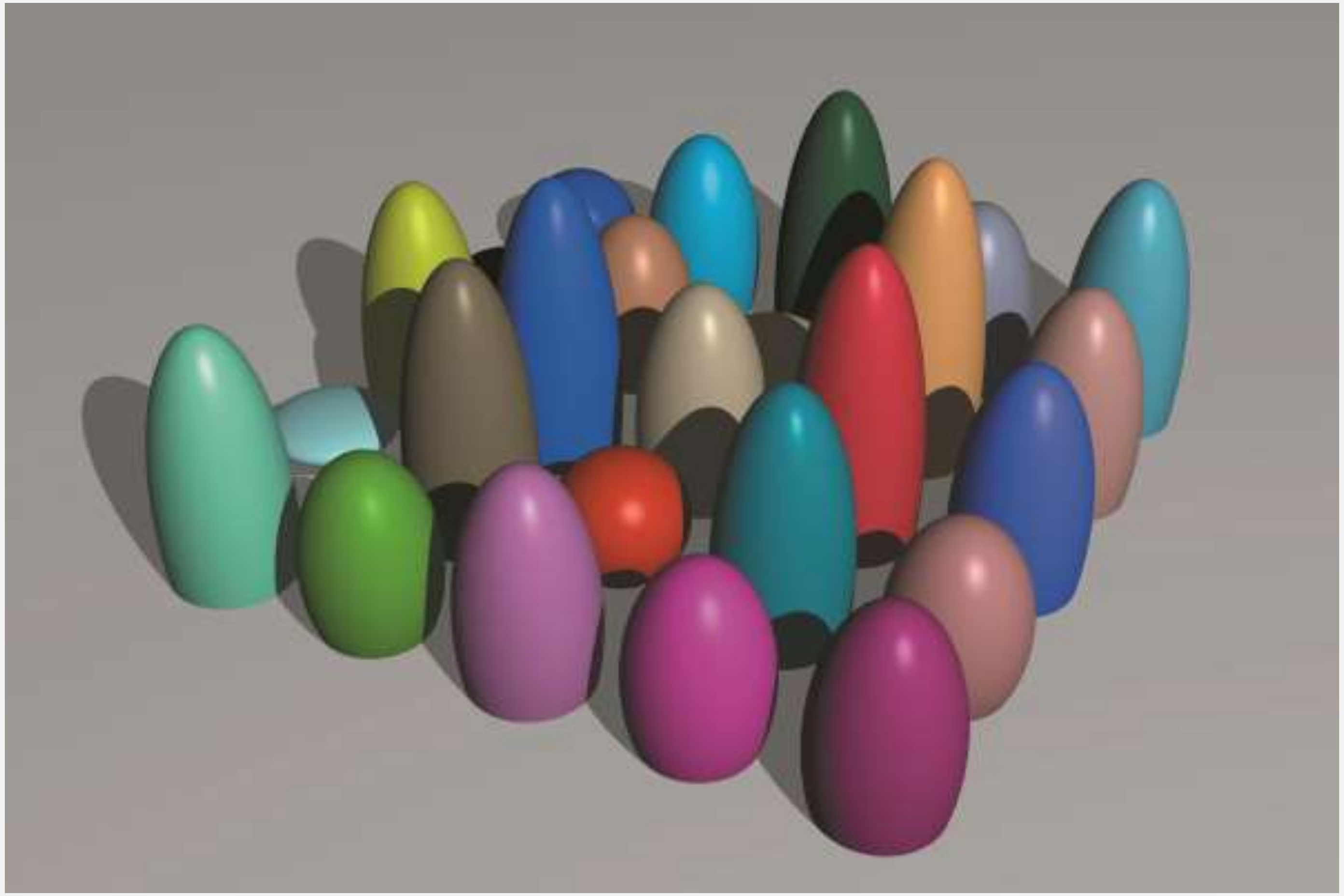
Quadrics - Example



[Suffern]

Outline

- Implicit surfaces
- Parametric surfaces
- Combined objects
- Triangles
- Axis-aligned boxes
- Isosurfaces in grids

Parametric Surfaces

- Are represented by functions with 2D parameters

$$x = f(u, v) \quad y = g(u, v) \quad z = h(u, v)$$

- Intersection is computed using a linear system with three equations and three unknowns *t*, *u*, *v*

$$o_x + td_x = f(u, v) \quad o_y + td_y = g(u, v) \quad o_z + td_z = h(u, v)$$

- Normal vector

$$\mathbf{n}(u, v) = \left(\frac{\partial f}{\partial u}, \frac{\partial g}{\partial u}, \frac{\partial h}{\partial u}\right) \times \left(\frac{\partial f}{\partial v}, \frac{\partial g}{\partial v}, \frac{\partial h}{\partial v}\right)$$

Parametric Surfaces, e.g., Cylinder, Sphere

- Cylinder about *z*-axis with parameters ϕ and ν

$$x = \cos\phi \qquad 0 \le \phi \le 2\pi$$
$$y = \sin\phi$$
$$z = z_{\min} + \nu(z_{\max} - z_{\min}) \quad 0 \le \nu \le 1$$

- Sphere centered at the origin with parameters ϕ and θ

$$x = \cos\phi\sin\theta \qquad 0 \le \phi \le 2\pi$$
$$y = \sin\phi\sin\theta \qquad 0 < \theta \le \pi$$
$$z = \cos\theta$$

- Parametric representations are used to render partial objects, e.g. $\phi_{\min} \le \phi \le \phi_{\max}$
- Can be combined with arbitrary affine transformations

Parametric Surfaces, e.g., Disk, Cone

– Disk with radius *r* at height *h* along the *z*-axis with inner radius r_i with parameters *u* and ν

$$\phi = u\phi_{\max} \qquad 0 \le u \le 1$$
$$x = ((1-\nu)r_i + \nu r)\cos\phi \qquad 0 \le \nu \le 1$$
$$y = ((1-\nu)r_i + \nu r)\sin\phi$$
$$z = h$$

– Cone with radius *r* and height *h* and parameters *u* and ν

$$\phi = u\phi_{\max} \qquad 0 \le u \le 1$$
$$x = r(1-\nu)\cos\phi \qquad 0 \le \nu \le 1$$
$$y = r(1-\nu)\sin\phi$$
$$z = \nu h$$

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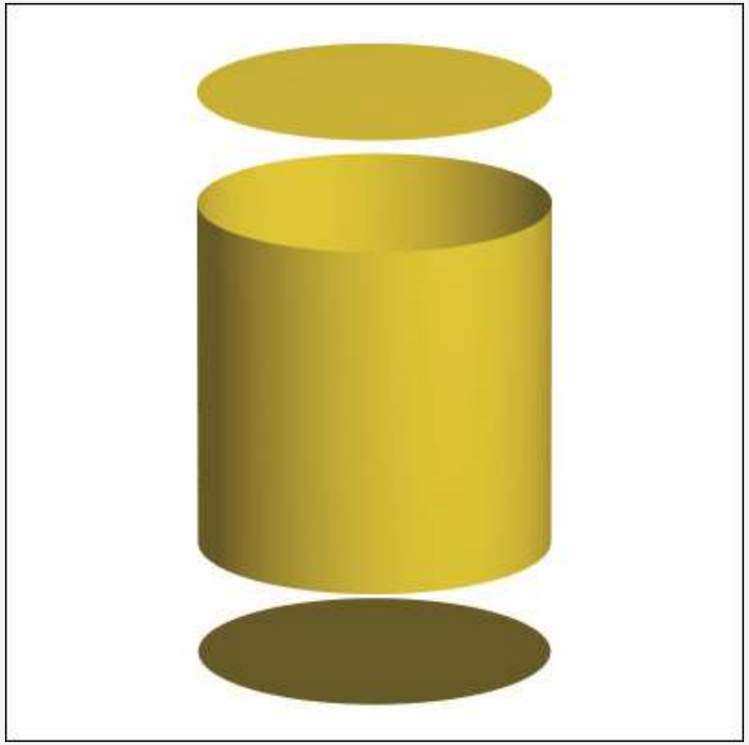
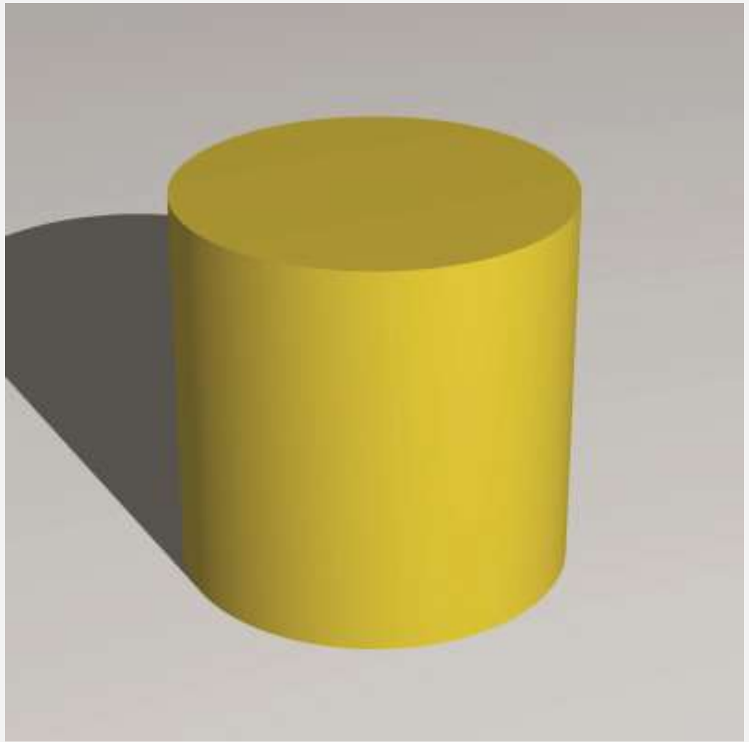
Compound Objects

– Consist of components





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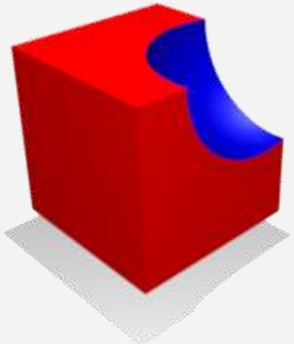
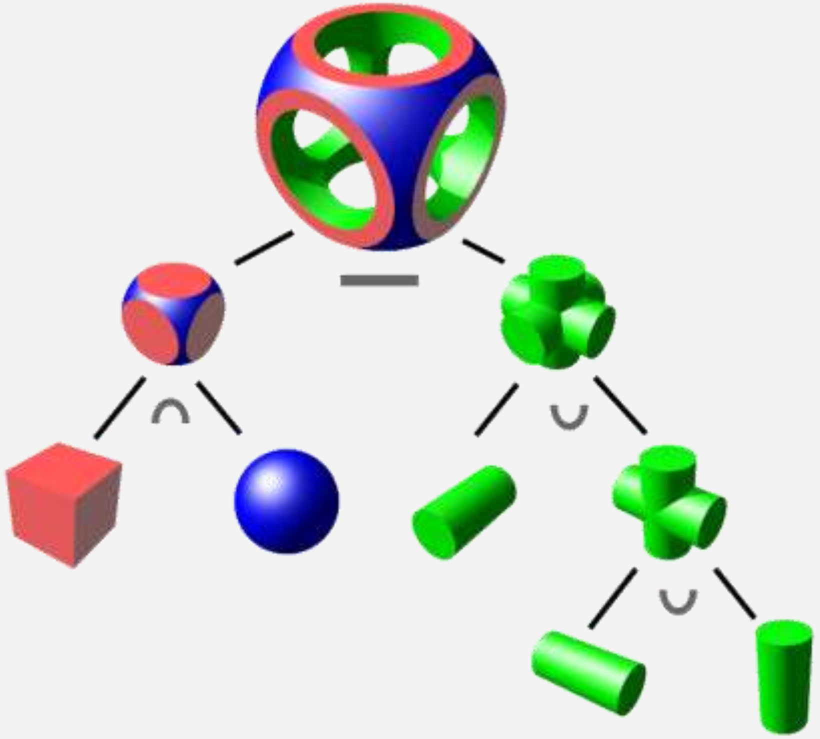
Constructive Solid Geometry

– Combine simple objects to complex geometry using Boolean operators



Difference of a cube and a sphere. Sphere intersections are only considered inside the cube. Cube intersections are not considered inside the sphere.

[Wikipedia: Constructive Solid Geometry]



[Wikipedia: Computergrafik]

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Triangle

- Parametric representation (barycentric coords)

$$\mathbf{p}(b_1, b_2) = (1 - b_1 - b_2)\mathbf{p_0} + b_1\mathbf{p_1} + b_2\mathbf{p_2}$$

$$b_1 \geq 0 \qquad b_2 \geq 0 \qquad b_1 + b_2 \leq 1$$

- Intersection is computed using a linear system

$$\mathbf{o} + t\mathbf{d} = (1 - b_1 - b_2)\mathbf{p_0} + b_1\mathbf{p_1} + b_2\mathbf{p_2}$$

- Solution (non-degenerated triangles, not parallel to ray)

$$\begin{pmatrix} t \\ b_1 \\ b_2 \end{pmatrix} = \frac{1}{(\mathbf{d}\times\mathbf{e_2})\cdot\mathbf{e_1}} \begin{pmatrix} (\mathbf{s}\times\mathbf{e_1})\cdot\mathbf{e_2} \\ (\mathbf{d}\times\mathbf{e_2})\cdot\mathbf{s} \\ (\mathbf{s}\times\mathbf{e_1})\cdot\mathbf{d} \end{pmatrix} \qquad \begin{aligned} \mathbf{e_1} &= \mathbf{p_1} - \mathbf{p_0} \\ \mathbf{e_2} &= \mathbf{p_2} - \mathbf{p_0} \\ \mathbf{s} &= \mathbf{o} - \mathbf{p_0} \end{aligned}$$

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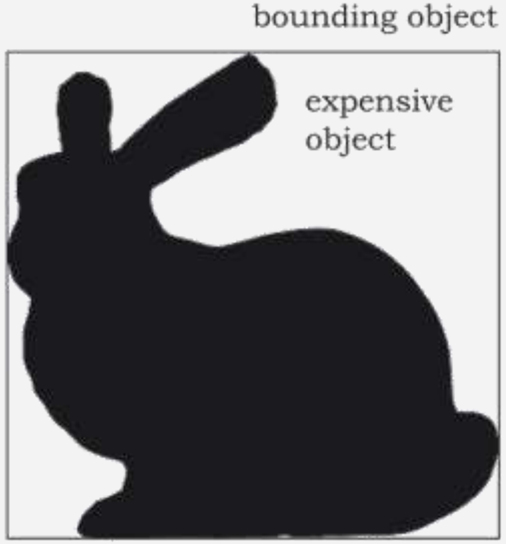
Axis-Aligned (Bounding) Box AABB

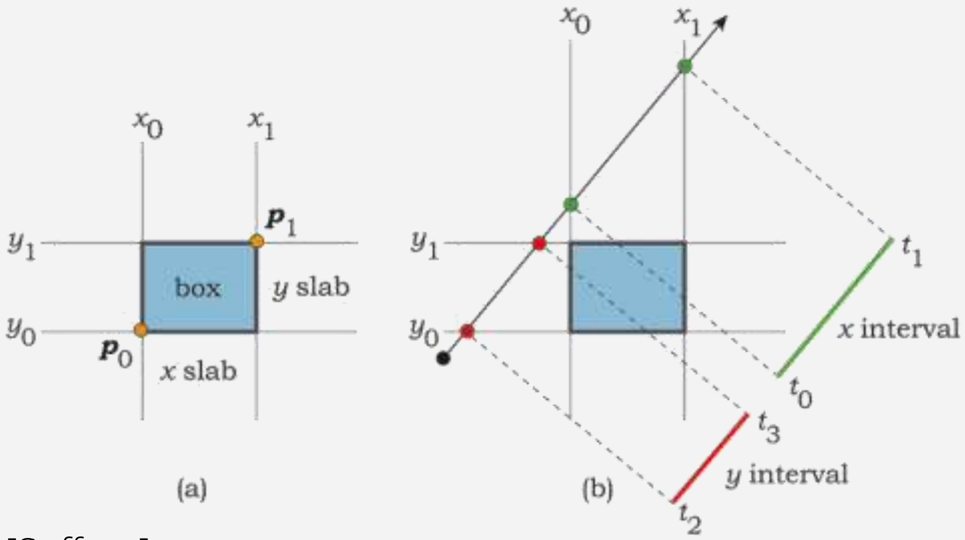
- Are commonly used to enclose complex geometry
- Accelerate the ray-object intersection
  - If a ray misses the simple box, the enclosed complex object cannot be hit by the ray
- AABBs are aligned with the principal axes of a coordinate system
  - To simplify intersection tests
  - To simplify the representation (two points represent three slabs in *x*-, *y*-, *z*-direction)



[Suffern]

Axis-Aligned (Bounding) Box AABB

- Boxes are represented by slabs
- Intersections of rays with slabs are analyzed to check for ray-box intersection
  - E.g. non-overlapping ray intervals within different slabs indicate that the ray misses the box



[Suffern]

$$\mathbf{n} \cdot (\mathbf{o} + t\mathbf{d} - \mathbf{r}) = 0$$

$$t = \frac{(\mathbf{r}-\mathbf{o})\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{d}}$$

general case

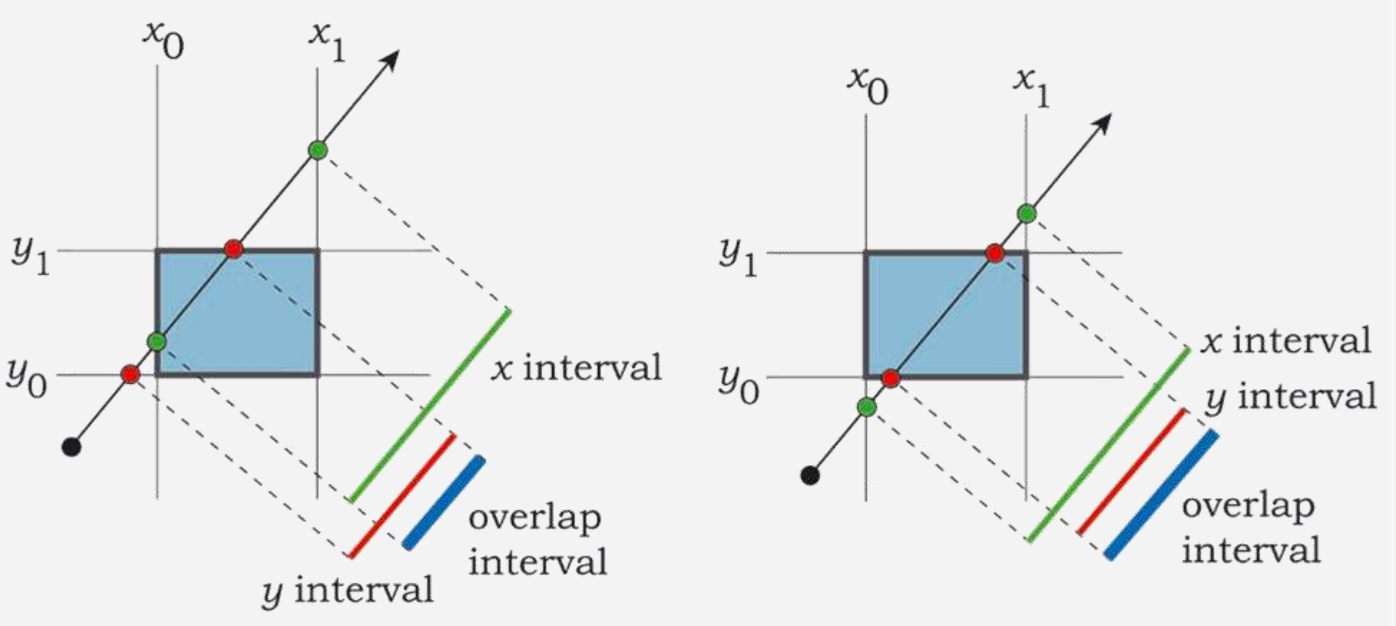
$$(1,0,0)^T \cdot (\mathbf{o} + t\mathbf{d} - (x_{0,1},0,0)^T) = 0$$

$$t_{x_{\min}, x_{\max}} = \frac{(x_{0,1} - o_x)}{d_x}$$

intersection with x-slab

Axis-Aligned (Bounding) Box AABB

– Overlapping ray intervals indicate intersections,
e.g. $t_{xmin} < t_{ymax} \wedge t_{xmax} > t_{ymin} \Rightarrow$ intersection
(largest entering value *t* is smaller than the smallest leaving value *t*,
only positive values t are considered)



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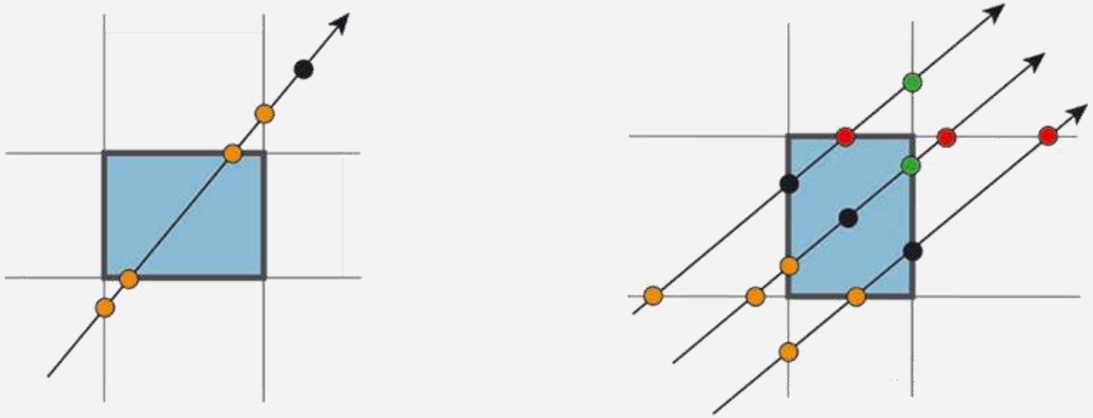
Axis-Aligned (Bounding) Box AABB

– Negative entering values and positive leaving values for all axes indicate that the ray starts inside the box



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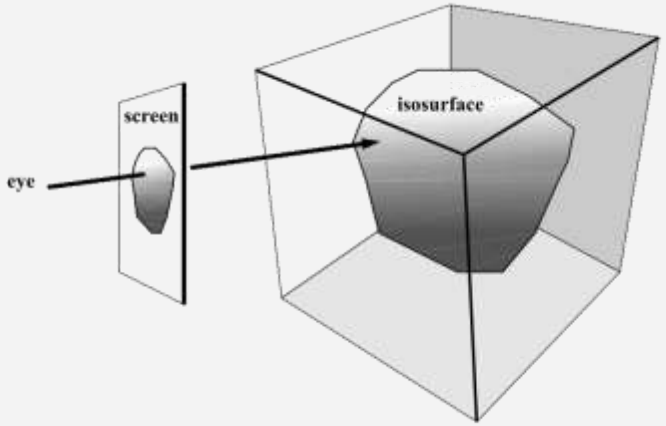
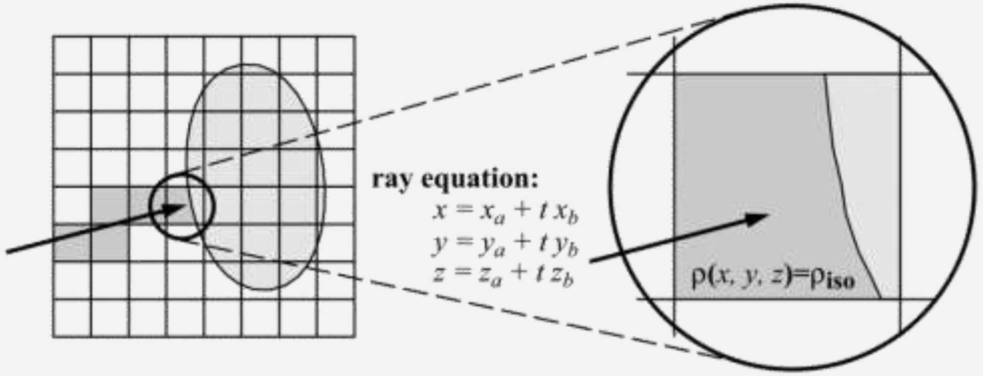
Setting



general setting

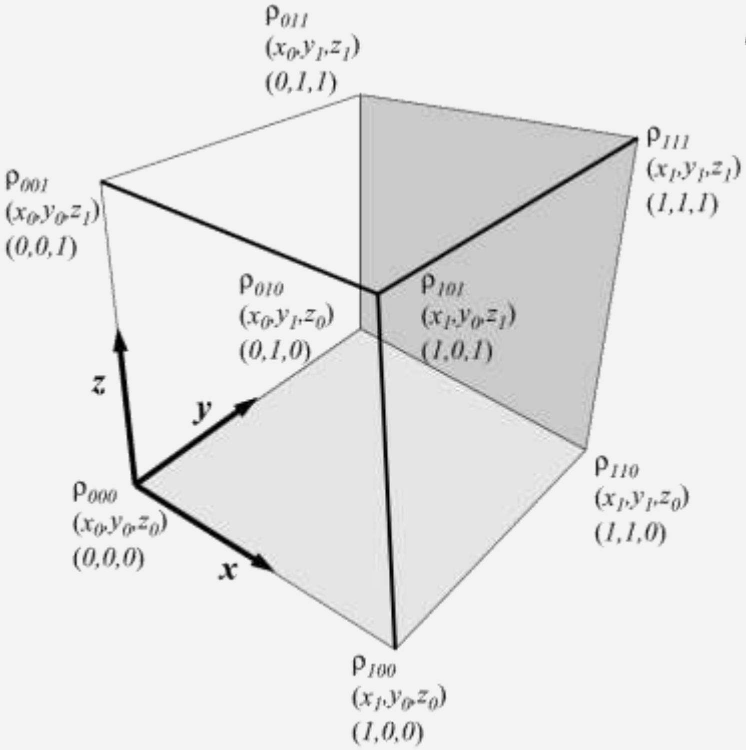


setting with a uniform grid

[Parker et al.]

Scalar-field Interpolation

– Trilinear interpolation of scalar values inside a grid cell



[Parker et al.]

$$
\begin{aligned}
\rho(u,v,w) &= (1-u)(1-v)(1-w)\rho_{000} + \\
&\quad (1-u)(1-v)(w)\rho_{001} + \\
&\quad (1-u)(v)(1-w)\rho_{010} + \\
&\quad (u)(1-v)(1-w)\rho_{100} + \\
&\quad (u)(1-v)(w)\rho_{101} + \\
&\quad (1-u)(v)(w)\rho_{011} + \\
&\quad (u)(v)(1-w)\rho_{110} + \\
&\quad (u)(v)(w)\rho_{111}
\end{aligned}
$$

$$
u = \frac{x - x_0}{x_1 - x_0} \qquad v = \frac{y - y_0}{y_1 - y_0} \qquad w = \frac{z - z_0}{z_1 - z_0}
$$

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Isosurface Normal

- Gradient of the scalar field

$$\mathbf{n} = \nabla\rho(x,y,z) = \left(\frac{\partial\rho(x,y,z)}{\partial x}, \frac{\partial\rho(x,y,z)}{\partial y}, \frac{\partial\rho(x,y,z)}{\partial z}\right)^T$$

- Approximated, e.g., with finite difference

$$n_x = \sum_{i,j,k=0,1} \frac{(-1)^{i+1} v_j w_k}{x_1 - x_0} \rho_{ijk}$$

$$n_y = \sum_{i,j,k=0,1} \frac{(-1)^{j+1} u_i w_k}{y_1 - y_0} \rho_{ijk}$$

$$n_z = \sum_{i,j,k=0,1} \frac{(-1)^{k+1} u_i v_j}{z_1 - z_0} \rho_{ijk}$$

[Parker et al.]

Ray-Isosurface Intersection

- Ray $\mathbf{r}(t) = \mathbf{o} + t\mathbf{d}$
- Isosurface intersection
  - Compute *t* with $\rho(\mathbf{r}(t)) = \rho_{iso} = \rho(\mathbf{o} + t\mathbf{d})$
- Cubic polynomial in *t*

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Application

- Rendering of surfaces in particle-based fluid simulations with large particle counts

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Summary

- If ray-object intersections and the surface normal can be computed, the object is renderable in a ray tracer
- Implicit surfaces
- Parametric surfaces
  - Used for partial objects
- Combinations of objects
- Triangles
- Axis-aligned boxes
  - Accelerated ray-object intersections for complex geometry