Advanced Computer Graphics
Ray-Object Intersections

Matthias Teschner
Rendering

- What is visible in an image?
  - Transformations from model space to screen space
  - Ray-object intersections

- Which color is it?
  - Shading/Lightning (e.g. Phong model)
  - Rendering equation

Ray Tracers
Compute ray-object intersections to estimate $q$ from $p$
Motivation

- Rays
  - A half-line specified by an origin / position \( \mathbf{o} \) and a direction \( \mathbf{d} \)
  - Parametric form \( \mathbf{r}(t) = \mathbf{o} + t\mathbf{d} \) with \( 0 \leq t \leq \infty \)
- Nearest intersection with all objects has to be computed, i.e. intersection with minimal \( t \geq 0 \)
- In implementations, usually \( t \geq \varepsilon \) to avoid self-intersections, e.g., if rays start at object surfaces
Outline

– Implicit surfaces
– Parametric surfaces
– Combined objects
– Triangles
– Axis-aligned boxes
– Isosurfaces in grids
Implicit Surfaces

- Implicit functions implicitly define a set of surface points
- For a surface point \((x,y,z)\), an implicit function \(f(x,y,z)\) is zero
- An intersection occurs, if a point on a ray satisfies the implicit equation \(f(x,y,z) = f(r(t)) = f(o + td) = 0\)
- E.g., all points \(p\) on a plane with surface normal \(n\) and offset \(r\) satisfy the equation \(n \cdot (p - r) = 0\)
- The intersection with a ray can be computed based on \(t\)

\[n \cdot (o + td - r) = 0 \quad t = \frac{(r-o) \cdot n}{n \cdot d} \quad \text{if } d \text{ is not orthogonal to } n\]
Implicit Surfaces - Normal

– Perpendicular to the surface
– Given by the gradient of the implicit function

$$\mathbf{n} = \nabla f(\mathbf{p}) = \left( \frac{\partial f(\mathbf{p})}{\partial x}, \frac{\partial f(\mathbf{p})}{\partial y}, \frac{\partial f(\mathbf{p})}{\partial z} \right)$$

– E.g., for a point $\mathbf{p}$ on a plane $f(\mathbf{p}) = \mathbf{n} \cdot (\mathbf{p} - \mathbf{r}) = 0$

$$\mathbf{n} = \nabla f(\mathbf{p}) = (n_x, n_y, n_z)$$
Quadrics

- E.g.
  - Sphere \( \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{a^2} - 1 = 0 \)
  - Ellipsoid \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \)
  - Paraboloid \( \frac{x^2}{a^2} + \frac{y^2}{a^2} - z = 0 \)
  - Hyperboloid \( \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} - 1 = 0 \)
  - Cone \( \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \)
  - Cylinder \( \frac{x^2}{a^2} + \frac{y^2}{a^2} - 1 = 0 \)

- Represented by quadratic equations, i.e. zero, one or two intersections with a ray
Quadrics - Sphere

– At the origin with radius one \( f(p) = x^2 + y^2 + z^2 - 1 = 0 \)
  \( (o_x + td_x)^2 + (o_y + td_y)^2 + (o_z + td_z)^2 - 1 = 0 \)

– Quadratic equation in \( t \)

\[
At^2 + Bt + C = 0 \quad A = d_x^2 + d_y^2 + d_z^2 \quad B = 2(d_x o_x + d_y o_y + d_z o_z) \\
C = o_x^2 + o_y^2 + o_z^2 - 1 \\
t_{0,1} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}
\]

– Surface normal

\( n = \nabla f(p) = (2x, 2y, 2z) \)

– The sphere can be combined with arbitrary affine transformations
Quadrics - Example
Outline

- Implicit surfaces
- Parametric surfaces
- Combined objects
- Triangles
- Axis-aligned boxes
- Isosurfaces in grids
Parametric Surfaces

- Are represented by functions with 2D parameters
  \[ x = f(u, v) \quad y = g(u, v) \quad z = h(u, v) \]

- Intersection is computed using a linear system with three equations and three unknowns \( t, u, v \)
  \[ o_x + td_x = f(u, v) \quad o_y + td_y = g(u, v) \quad o_z + td_z = h(u, v) \]

- Normal vector
  \[ \mathbf{n}(u, v) = \left( \frac{\partial f}{\partial u}, \frac{\partial g}{\partial u}, \frac{\partial h}{\partial u} \right) \times \left( \frac{\partial f}{\partial v}, \frac{\partial g}{\partial v}, \frac{\partial h}{\partial v} \right) \]
Parametric Surfaces, e.g., Cylinder, Sphere

- Cylinder about z-axis with parameters $\phi$ and $\nu$
  \[ x = \cos \phi \quad 0 \leq \phi \leq 2\pi \]
  \[ y = \sin \phi \]
  \[ z = z_{\text{min}} + \nu(z_{\text{max}} - z_{\text{min}}) \quad 0 \leq \nu \leq 1 \]

- Sphere centered at the origin with parameters $\phi$ and $\theta$
  \[ x = \cos \phi \sin \theta \quad 0 \leq \phi \leq 2\pi \]
  \[ y = \sin \phi \sin \theta \quad 0 < \theta \leq \pi \]
  \[ z = \cos \theta \]

- Parametric representations are used to render partial objects, e.g. $\phi_{\text{min}} \leq \phi \leq \phi_{\text{max}}$

- Can be combined with arbitrary affine transformations
**Parametric Surfaces, e.g., Disk, Cone**

– Disk with radius $r$ at height $h$ along the $z$-axis with inner radius $r_i$ with parameters $u$ and $v$

\[
\begin{align*}
\phi &= u \phi_{\text{max}} \\
x &= ((1 - v)r_i + vr) \cos \phi \\
y &= ((1 - v)r_i + vr) \sin \phi \\
z &= h
\end{align*}
\]

– Cone with radius $r$ and height $h$ and parameters $u$ and $v$

\[
\begin{align*}
\phi &= u \phi_{\text{max}} \\
x &= r(1 - v) \cos \phi \\
y &= r(1 - v) \sin \phi \\
z &= vh
\end{align*}
\]
Outline

– Implicit surfaces
– Parametric surfaces
– Combined objects
– Triangles
– Axis-aligned boxes
– Isosurfaces in grids
Compound Objects

– Consist of components

[Suffern]
Constructive Solid Geometry

– Combine simple objects to complex geometry using Boolean operators

Difference of a cube and a sphere. Sphere intersections are only considered inside the cube. Cube intersections are not considered inside the sphere.

[Wikipedia: Constructive Solid Geometry]  [Wikipedia: Computergrafik]

University of Freiburg – Computer Science Department – 16
Outline

– Implicit surfaces
– Parametric surfaces
– Combined objects
– **Triangles**
– Axis-aligned boxes
– Isosurfaces in grids
Triangle

- Parametric representation (barycentric coords)
  \[ p(b_1, b_2) = (1 - b_1 - b_2)p_0 + b_1p_1 + b_2p_2 \]
  \[ b_1 \geq 0 \quad b_2 \geq 0 \quad b_1 + b_2 \leq 1 \]

- Intersection is computed using a linear system
  \[ o + td = (1 - b_1 - b_2)p_0 + b_1p_1 + b_2p_2 \]

- Solution (non-degenerated triangles, not parallel to ray)
  \[
  \begin{pmatrix} t \\
  b_1 \\
  b_2 
  \end{pmatrix} = \frac{1}{(d \times e_2) \cdot e_1} \begin{pmatrix} (s \times e_1) \cdot e_2 \\
  (d \times e_2) \cdot s \\
  (s \times e_1) \cdot d 
  \end{pmatrix}
  \]
  \[ e_1 = p_1 - p_0 \]
  \[ e_2 = p_2 - p_0 \]
  \[ s = o - p_0 \]
Outline

- Implicit surfaces
- Parametric surfaces
- Combined objects
- Triangles
- Axis-aligned boxes
- Isosurfaces in grids
Axis-Aligned (Bounding) Box AABB

– Are commonly used to enclose complex geometry
– Accelerate the ray-object intersection
  – If a ray misses the simple box, the enclosed complex object cannot be hit by the ray
– AABBs are aligned with the principal axes of a coordinate system
  – To simplify intersection tests
  – To simplify the representation (two points represent three slabs in x-, y-, z-direction)
Axis-Aligned (Bounding) Box AABB

- Boxes are represented by slabs
- Intersections of rays with slabs are analyzed to check for ray-box intersection
  - E.g. non-overlapping ray intervals within different slabs indicate that the ray misses the box

\[ \mathbf{n} \cdot (\mathbf{o} + td - \mathbf{r}) = 0 \]
\[ t = \frac{(\mathbf{r} - \mathbf{o}) \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{d}} \]
\[ (1, 0, 0)^T \cdot (\mathbf{o} + td - (x_{0,1}, 0, 0)^T) = 0 \]
\[ t_{x_{\min}, x_{\max}} = \frac{(x_{0,1} - o_x)}{d_x} \]
Axis-Aligned (Bounding) Box AABB

- Overlapping ray intervals indicate intersections,
  e.g. \( t_{x_{\text{min}}} < t_{y_{\text{max}}} \land t_{x_{\text{max}}} > t_{y_{\text{min}}} \Rightarrow \text{intersection} \)
  (largest entering value \( t \) is smaller than the smallest leaving value \( t \),
  only positive values \( t \) are considered)
Axis-Aligned (Bounding) Box AABB

– Negative entering values and positive leaving values for all axes indicate that the ray starts inside the box
Outline

– Implicit surfaces
– Parametric surfaces
– Combined objects
– Triangles
– Axis-aligned boxes
– Isosurfaces in grids
Setting

- General setting
- Setting with a uniform grid
Scalar-field Interpolation

– Trilinear interpolation of scalar values inside a grid cell

\[
\rho(u, v, w) = (1 - u)(1 - v)(1 - w)\rho_{000} + \\
(1 - u)(1 - v)(w)\rho_{001} + \\
(1 - u)(v)(1 - w)\rho_{010} + \\
(u)(1 - v)(1 - w)\rho_{100} + \\
(u)(1 - v)(w)\rho_{101} + \\
(1 - u)(v)(w)\rho_{011} + \\
(u)(v)(1 - w)\rho_{110} + \\
(u)(v)(w)\rho_{111}
\]

\[
u = \frac{x - x_0}{x_1 - x_0}, \quad v = \frac{y - y_0}{y_1 - y_0}, \quad w = \frac{z - z_0}{z_1 - z_0}
\]
Isosurface Normal

– Gradient of the scalar field

\[ \mathbf{n} = \nabla \rho(x, y, z) = \left( \frac{\partial \rho(x, y, z)}{\partial x}, \frac{\partial \rho(x, y, z)}{\partial y}, \frac{\partial \rho(x, y, z)}{\partial z} \right)^T \]

– Approximated, e.g., with finite difference

\[ n_x = \sum_{i,j,k=0,1} \frac{(-1)^{i+1} v_j w_k}{x_1 - x_0} \rho_{ijk} \]

\[ n_y = \sum_{i,j,k=0,1} \frac{(-1)^{j+1} u_i w_k}{y_1 - y_0} \rho_{ijk} \]

\[ n_z = \sum_{i,j,k=0,1} \frac{(-1)^{k+1} u_i v_j}{z_1 - z_0} \rho_{ijk} \]
Ray-Isosurface Intersection

- Ray $\mathbf{r}(t) = \mathbf{o} + t\mathbf{d}$
- Isosurface intersection
  - Compute $t$ with $\rho(\mathbf{r}(t)) = \rho_{iso} = \rho(\mathbf{o} + t\mathbf{d})$
- Cubic polynomial in $t$
Application

– Rendering of surfaces in particle-based fluid simulations with large particle counts
Summary

- If ray-object intersections and the surface normal can be computed, the object is renderable in a ray tracer.
- Implicit surfaces
- Parametric surfaces
  - Used for partial objects
- Combinations of objects
- Triangles
- Axis-aligned boxes
  - Accelerated ray-object intersections for complex geometry