Advanced Computer Graphics

Aliasing

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Outline

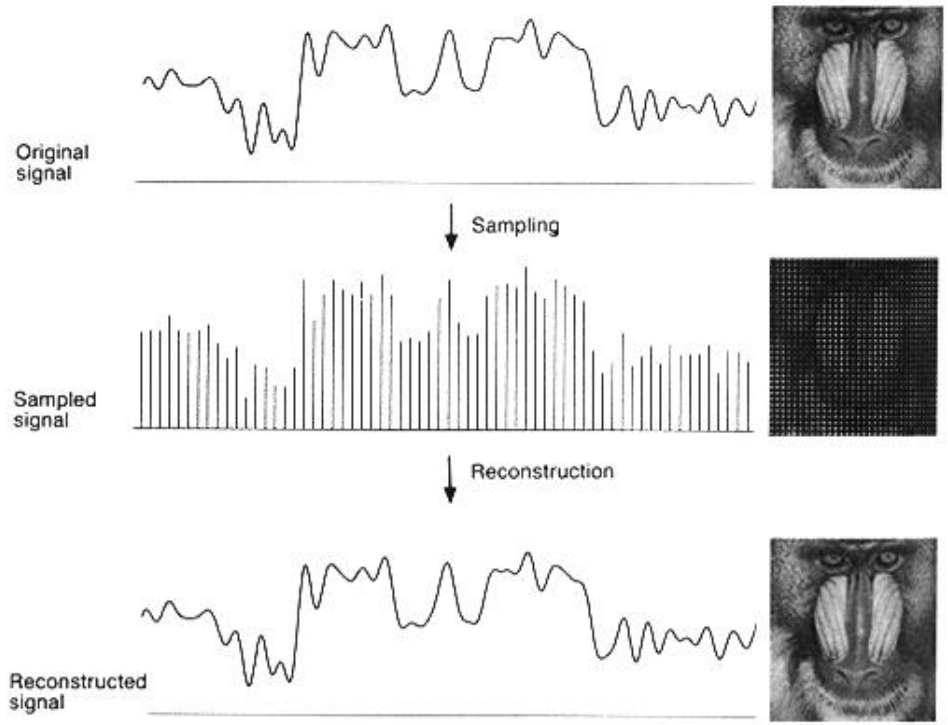
- motivation
- Fourier analysis
- filtering
- sampling
- reconstruction / aliasing
- antialiasing

Motivation

- incident radiance is a continuous function over the image plane
- ray tracing samples this function at discrete positions
- discrete samples
  - only approximately represent / reconstruct the original function
  - can introduce artifacts (aliasing), e.g. missing details, staircase artifacts / jaggies, erroneous patterns (Moire effect)
- goal
  - choose appropriate samples to represent the original function as good as possible

Motivation

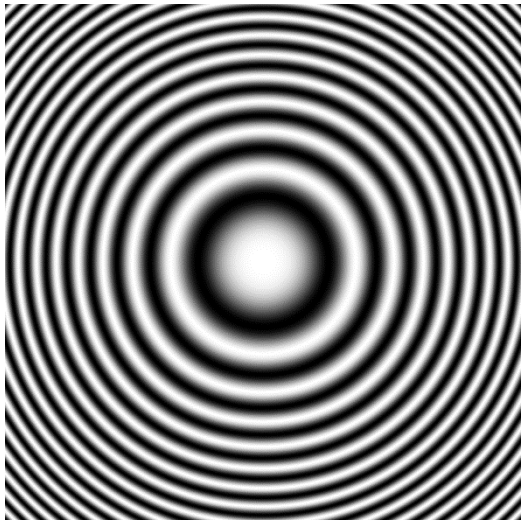
- sampling and reconstruction
- inappropriate sampling can cause artifacts in reconstructed functions



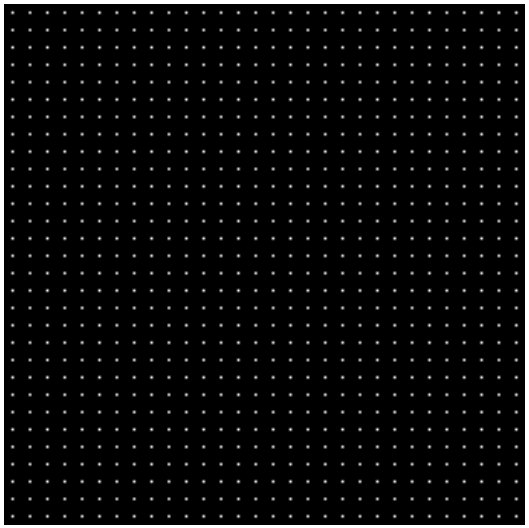
[Foley, van Dam, Feiner, Huges]

Motivation

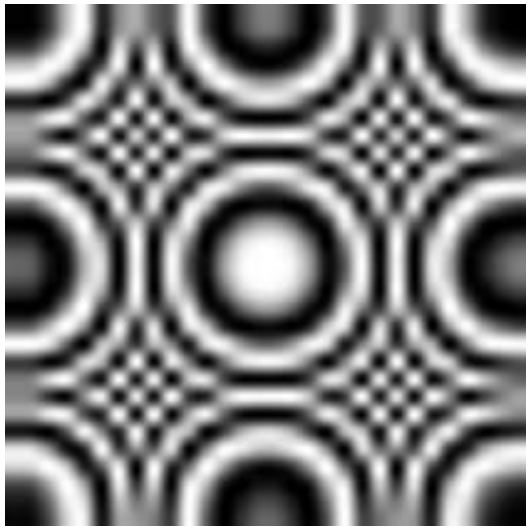
- aliasing artifacts, e.g. Moire pattern



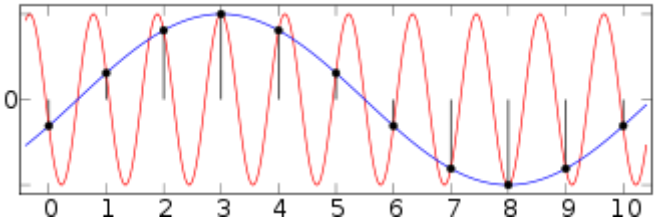
original signal



sampled signal



reconstructed signal



red - original signal

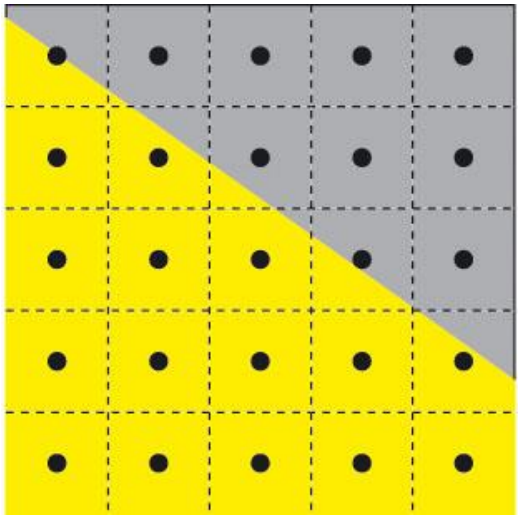
black dots - samples

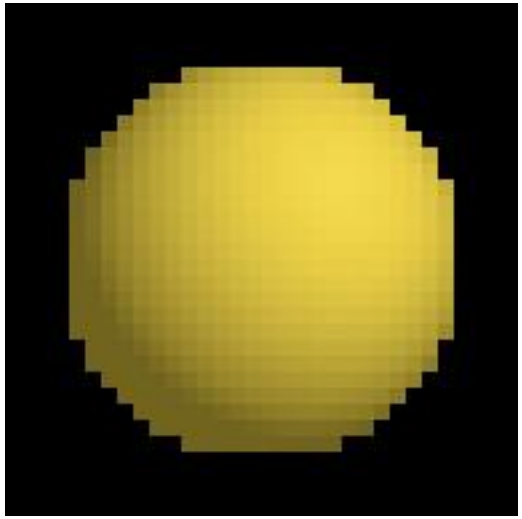
blue - reconstructed signal

[Wikipedia: Alias-Effekt]

Motivation

- aliasing artifacts, e.g. jaggies

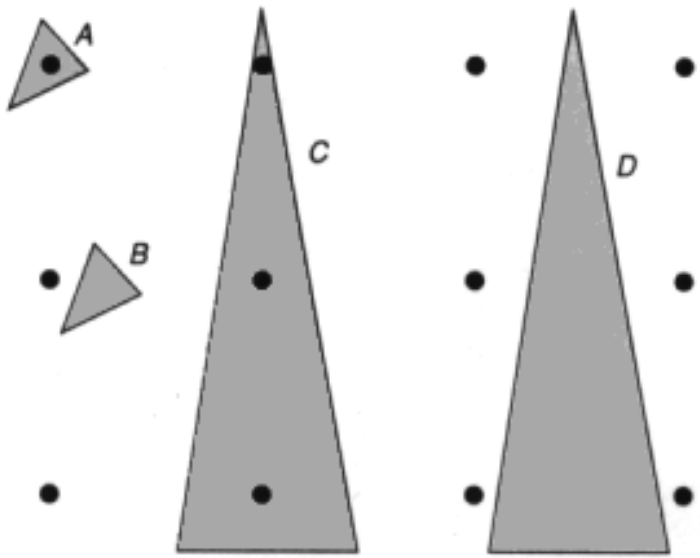




[Suffern]

Motivation

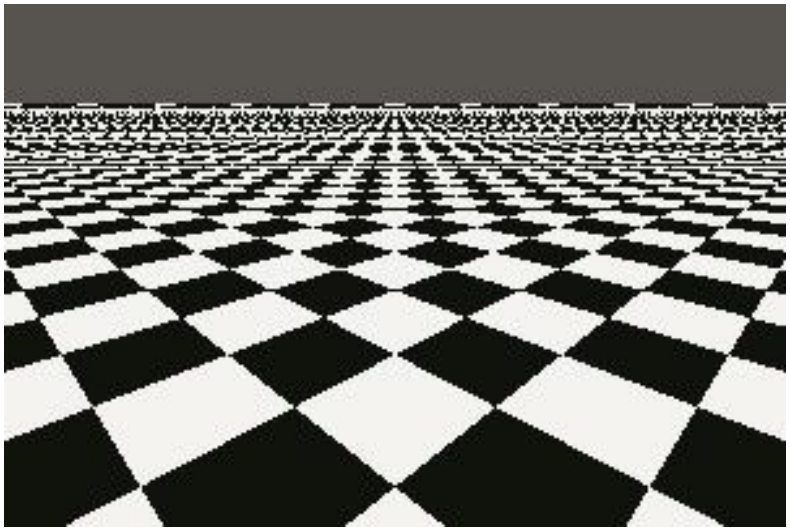
- aliasing artifacts, e.g. missing details



[Foley, van Dam, Feiner, Huges]

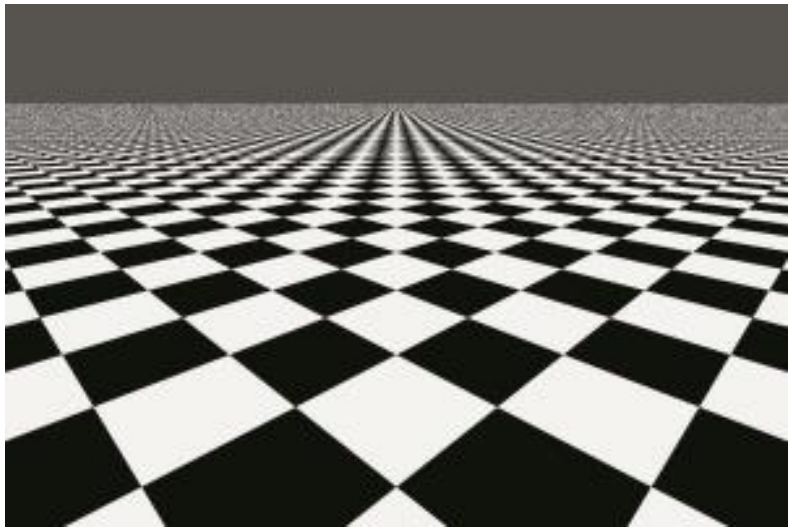
Motivation

- antialiasing
  - reduction of erroneous patterns



[Suffern]

aliasing artifacts



regular artifacts replaced
by less disturbing noise

Outline

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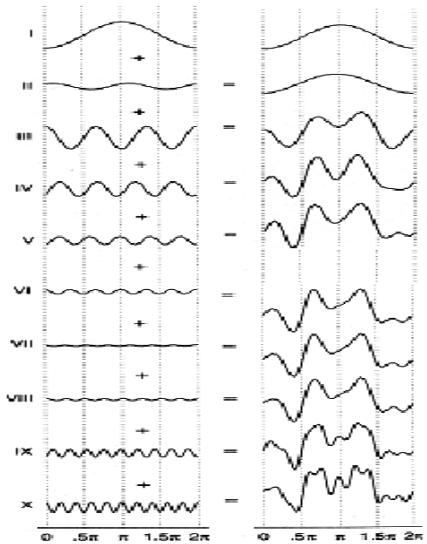
Spectrum of a Function

- Fourier transform
  - decomposes a function into weighted sum of shifted sinusoids
  - computes amplitude and phase shift of frequencies contained in the function
  - transforms from the spatial domain to the frequency domain



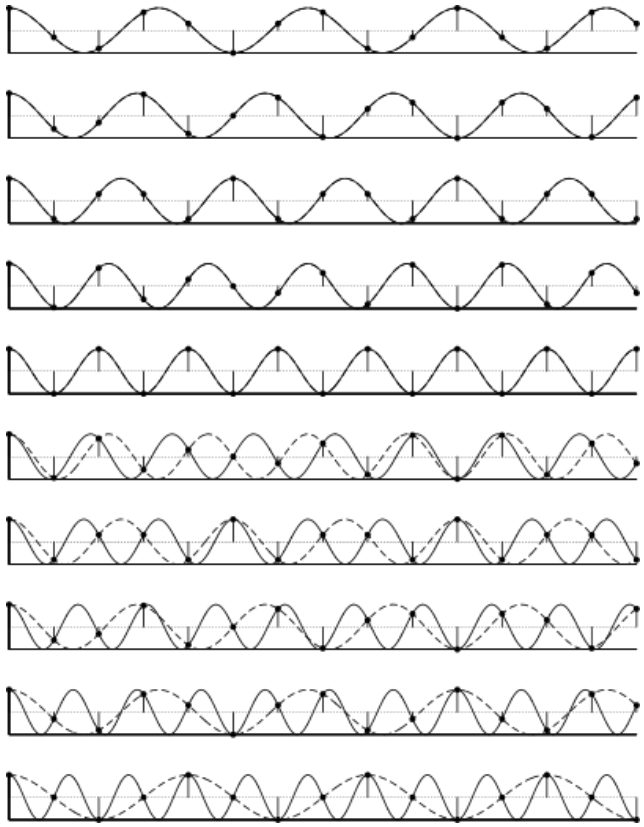
$$\mathfrak{F}\{f(x)\} = F(\omega)$$

$$\mathfrak{F}^{-1}\{F(\omega)\} = f(x)$$

[Foley, van Dam, Feiner, Huges]

Spectrum of a Function - Motivation

- analysis in the frequency domain allows to understand aliasing
- aliasing is then reduced by
  - adapting the sampling
  - filtering the original signal (for textures)



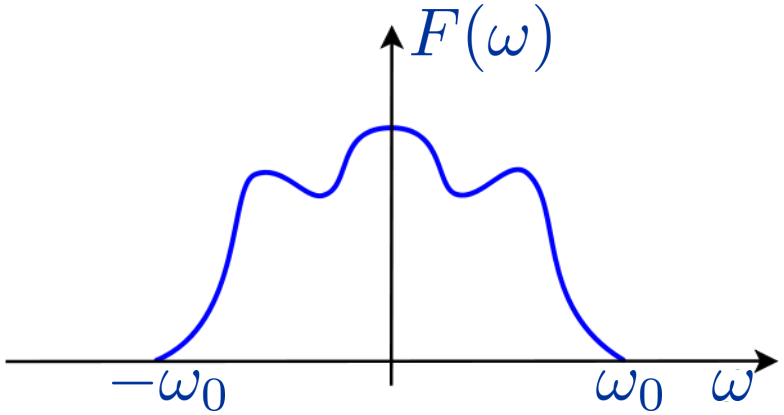
appropriate sampling frequencies

inappropriate sampling frequencies

[Wikipedia: Nyquist-Shannon-Abtasttheorem]

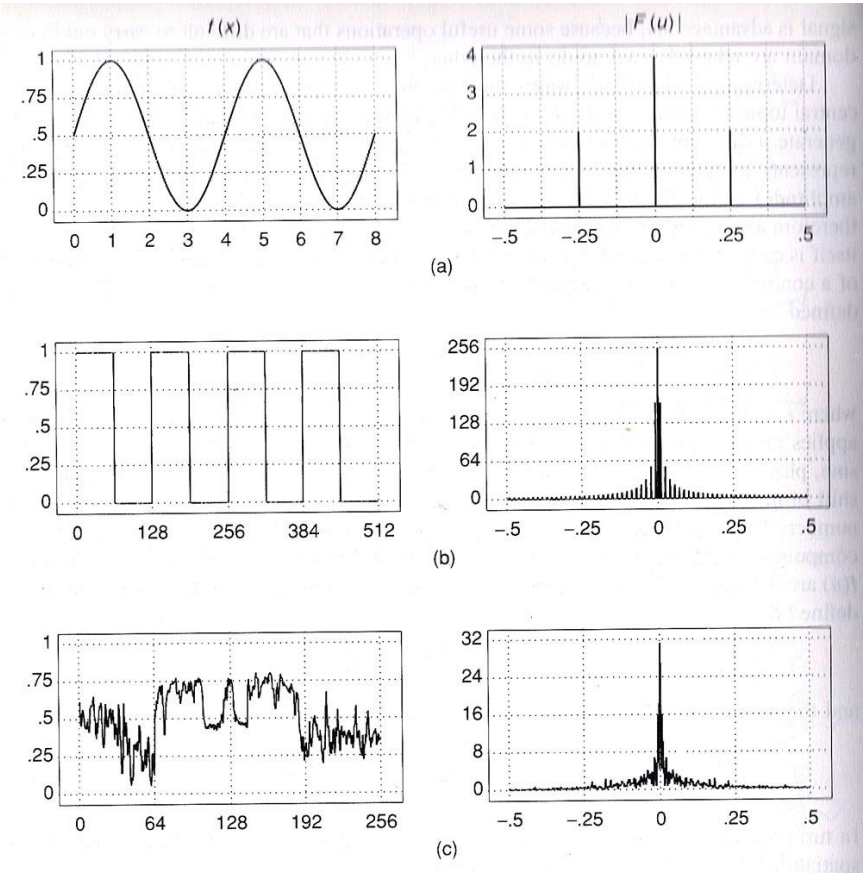
Spectrum of a Function - Motivation

- sampling and reconstruction can be analyzed in the frequency domain
- a band-limited function with $F(\omega) = 0$ for all $\omega > \omega_0$ has to be sampled with a frequency $\omega_{\text{sampling}} > 2\omega_0$ in order to be able to reconstruct the original function from the samples (Nyquist-Shannon sampling theorem)
- Nyquist frequency ω_0
- Nyquist rate ω_{sampling}



Fourier Transform - Examples

- signals in spatial and frequency domain



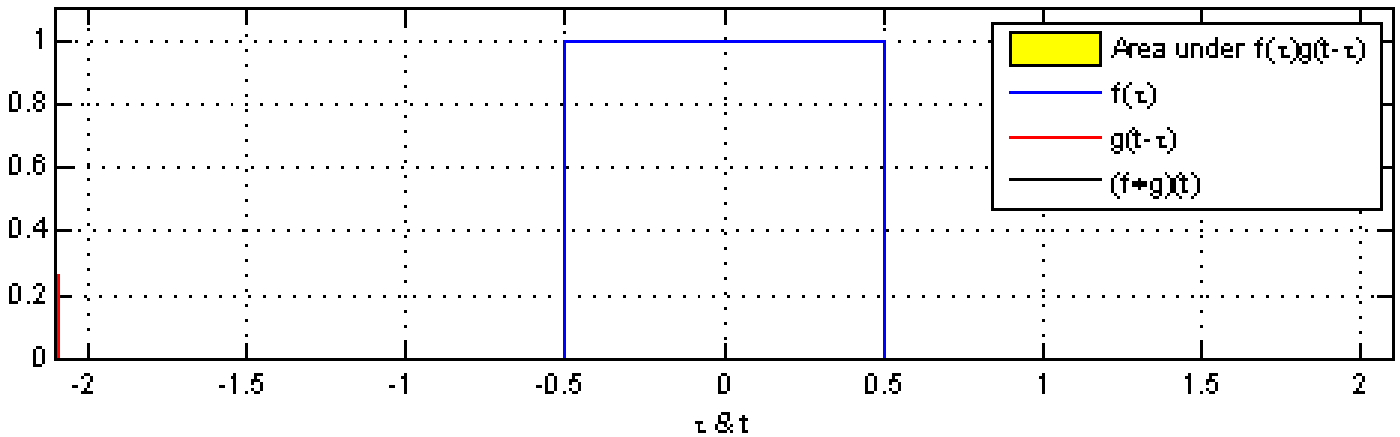
[Foley, van Dam, Feiner, Huges]

Fourier Transform - Properties

- Fourier transform of the product of two functions is equivalent to the convolution of the individual Fourier transforms
  $\mathfrak{F}\{f(x)g(x)\} = F(\omega) \otimes G(\omega)$
- convolution in the spatial domain is equivalent to multiplication in the frequency domain
  $\mathfrak{F}\{f(x) \otimes g(x)\} = F(\omega)G(\omega)$
- important in understanding how filtering and reconstruction affect the spectrum of a function

Convolution

- $(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau$
- convolution computes a weighted average of f using the weighting kernel g



[Wikipedia: Convolution]

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Low-Pass Filtering

- convolution is used to filter and reconstruct functions

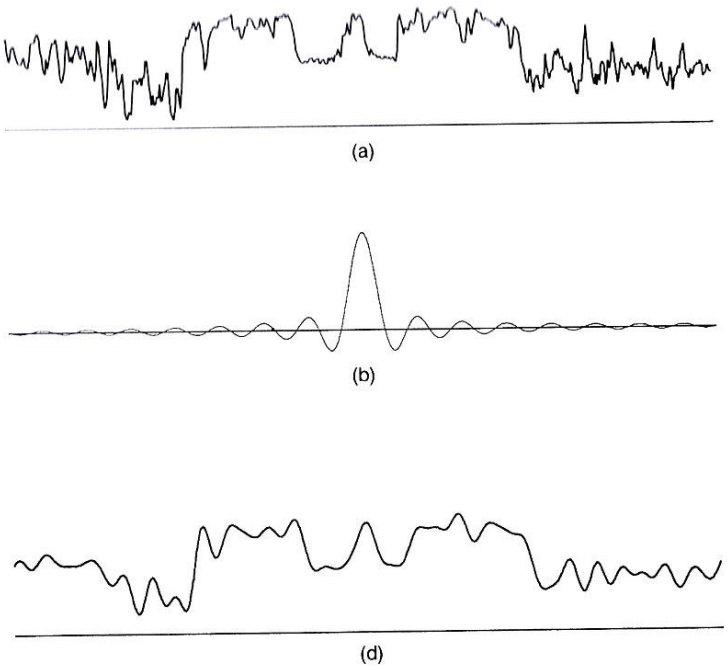


Fig. 14.23 Low-pass filtering in the spatial domain. (a) Original signal. (b) Sinc filter. (c) Signal with filter, with value of filtered signal shown as a black dot (●) at filter's origin. (d) Filtered signal. (Courtesy of George Wolberg, Columbia University.)

original function

sinc function

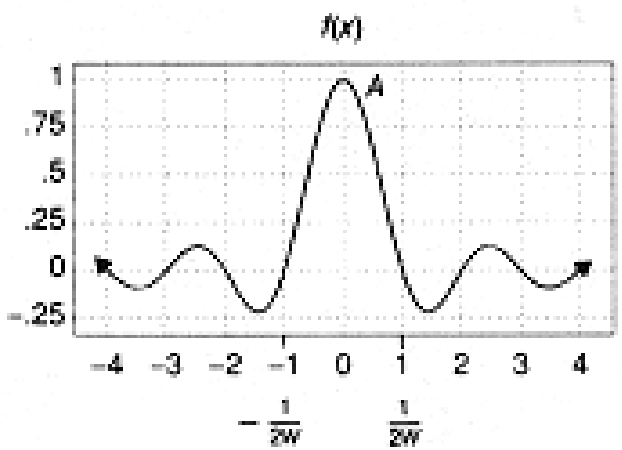
$$f(x) = \frac{\sin(x)}{x}$$

band-limited signal

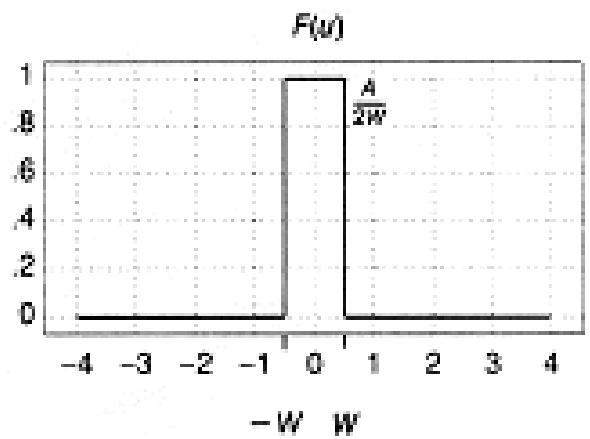
[Foley, van Dam, Feiner, Huges]

Low-Pass Filtering

- sinc function in spatial and frequency domain



sinc function

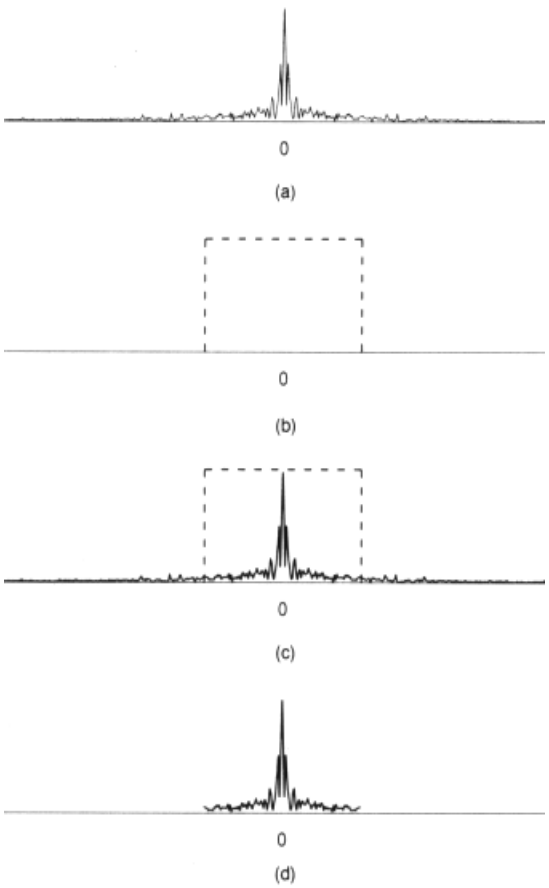


box function

[Foley, van Dam, Feiner, Huges]

Low-Pass Filtering

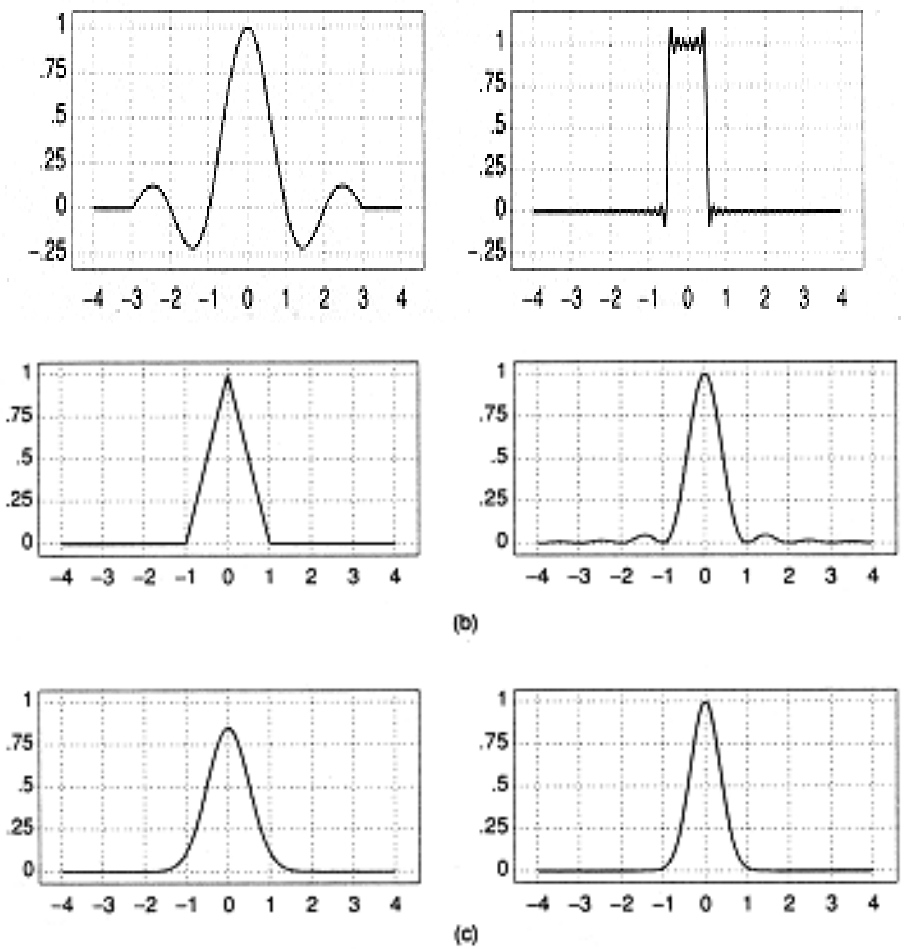
- convolution with *sinc* function in the spatial domain corresponds to multiplication with a box function in the frequency domain
- given a sampling rate, this low-pass filter completely suppresses all frequency components above the Nyquist frequency
  - aliasing is avoided in the reconstruction process
- applied in texturing
- in ray tracing, the original function cannot be simply filtered



[Foley, van Dam, Feiner, Huges]

Approximate Low-Pass Filtering

- truncated sinc
  - Gibbs phenomenon
- triangle
- Gaussian



[Foley, van Dam, Feiner, Huges]

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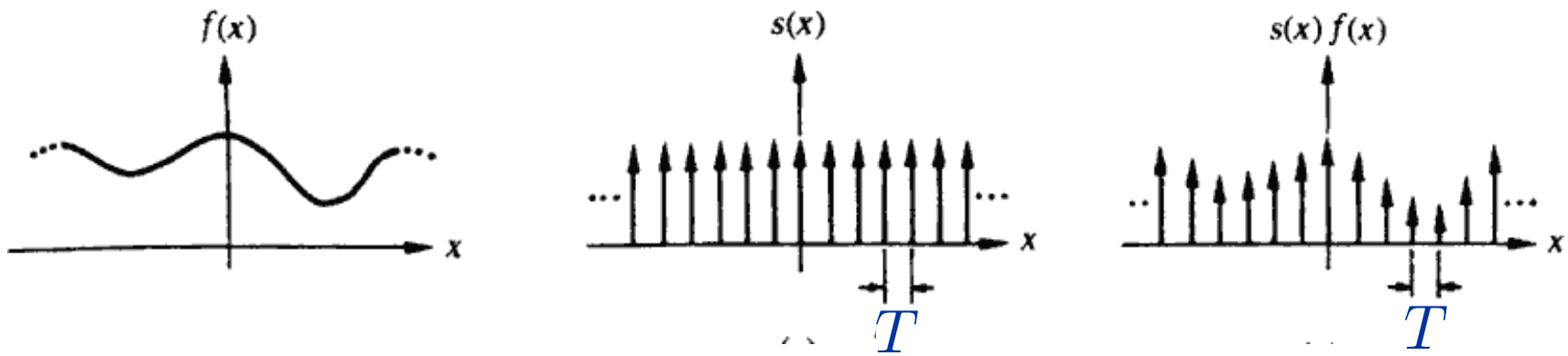
Sampling

- sampling a function corresponds to multiplying it in the spatial domain by a Dirac comb function
- $\mathrm{III}_T(x) = \sum_{i=-\infty}^{\infty} \delta(x - iT)$



original function

Dirac comb function
(shah function)

sampled function

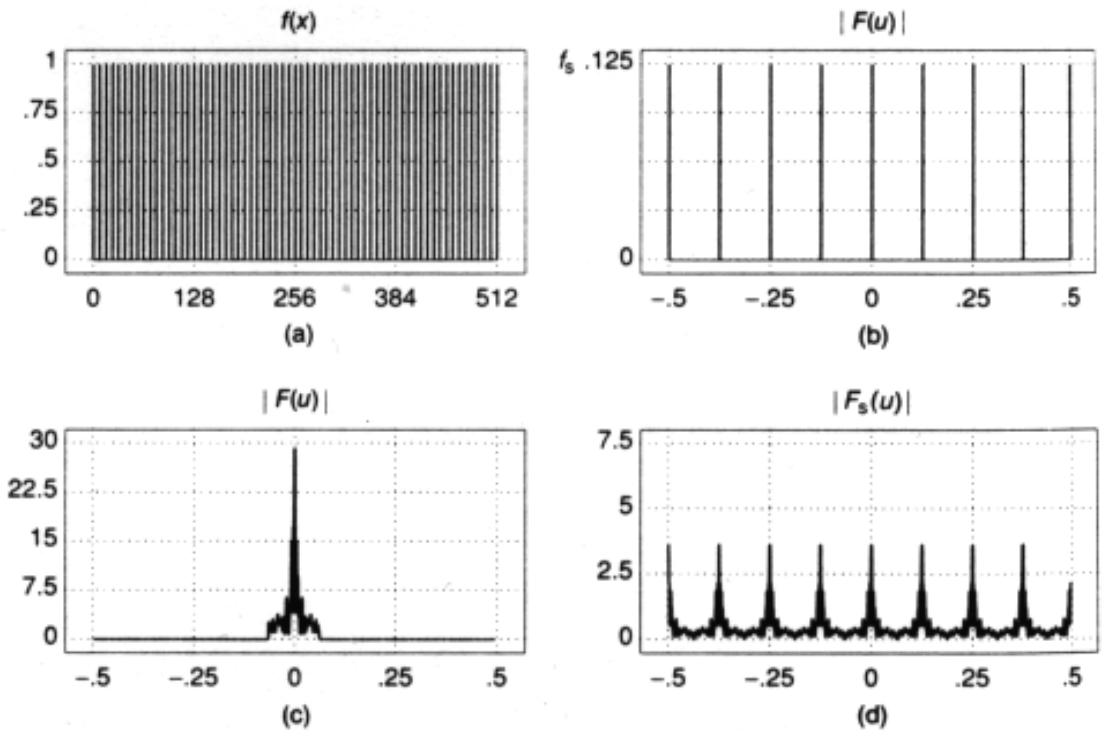
[Foley, van Dam, Feiner, Huges]

Sampling

- in frequency domain, sampling is a convolution of the function's spectrum with a Dirac comb function

Dirac comb in the spatial domain

$$\mathrm{III}_T(x)$$



Dirac comb in the frequency domain

$$\mathrm{III}_{1/T}(\omega)$$

spectrum of the function

$$F(\omega)$$

spectrum of the sampled function

$$F(\omega) \otimes \mathrm{III}_{1/T}(\omega)$$

[Foley, van Dam, Feiner, Huges]

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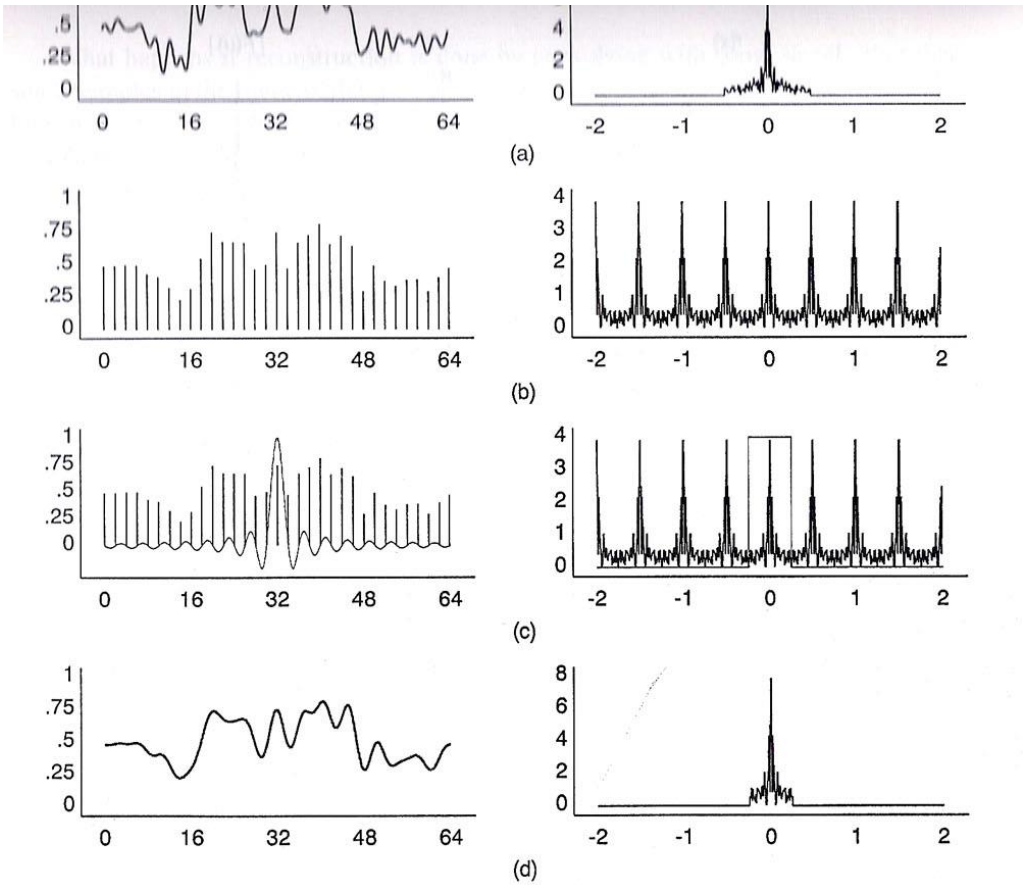
Reconstruction

- as a result of sampling, F contains an infinite number of replications of the spectrum of the original function f at multiples of the sampling frequency
- reconstruction tries to remove all but the spectrum of the original function by multiplying with a box filter in the frequency domain (corresponding to a convolution of the sampled function with sinc in the spatial domain)
- our visual system reconstructs a function from pixel values
- in ray tracing, incident radiance at pixels can be reconstructed from several samples

Reconstruction

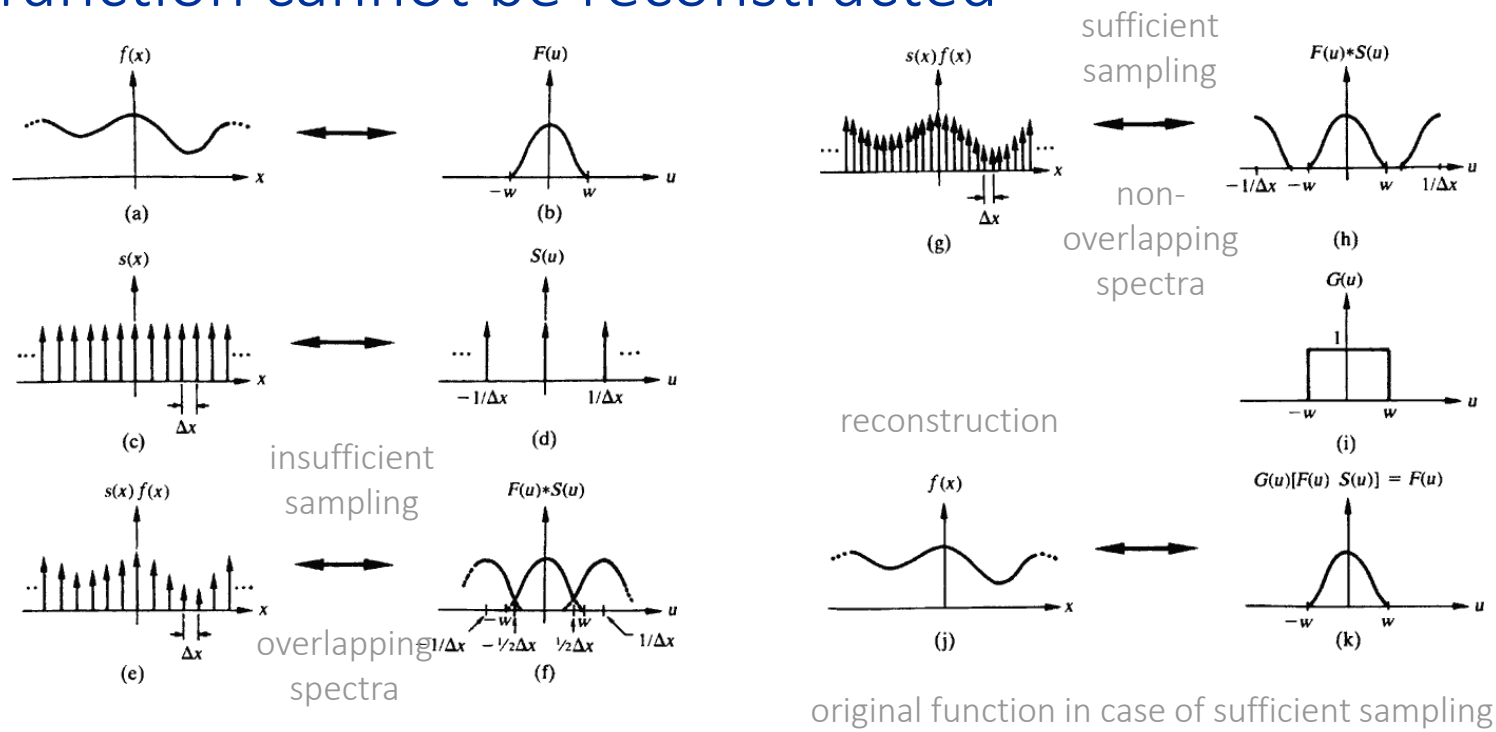
- sampling and reconstruction in spatial and frequency domain



[Foley, van Dam, Feiner, Huges]

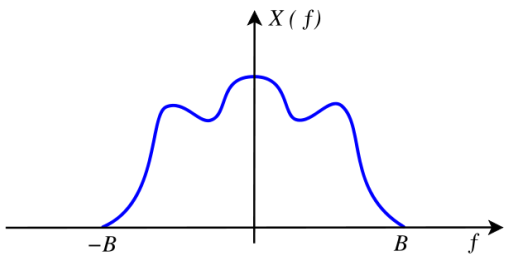
Aliasing

- if the sampling frequency is too low, the replicated copies of the spectra overlap and the spectrum of the original function cannot be reconstructed

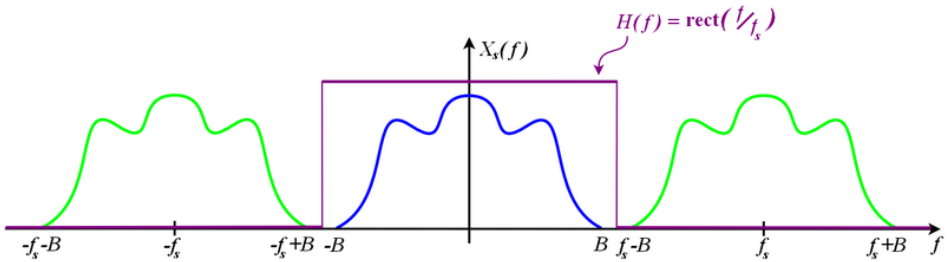


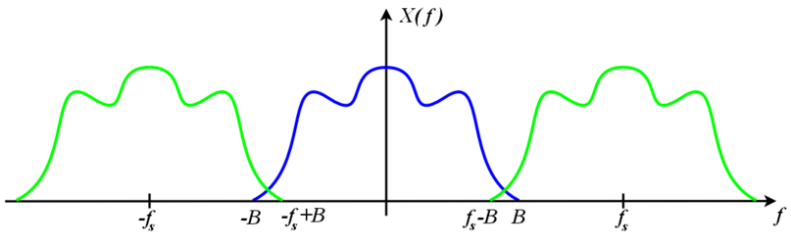
[Foley, van Dam, Feiner, Huges]

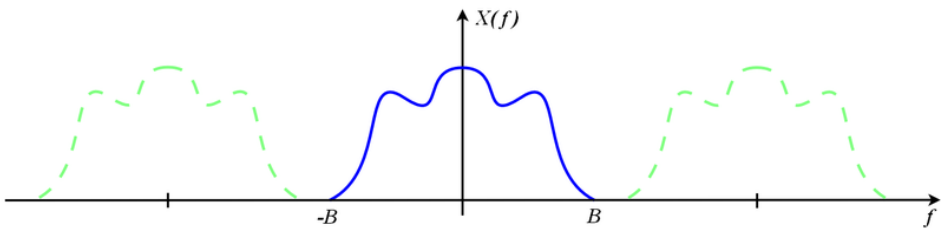
Aliasing



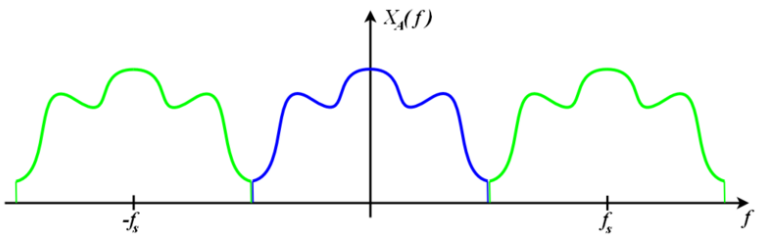
spectrum of a band-limited function







spectrum of sufficiently sampled band-limited function, a box filter can reconstruct the original spectrum



spectrum of an insufficiently sampled band-limited function, the original spectrum cannot be reconstructed

[Wikipedia: Nyquist-Shannon-Abtasttheorem]

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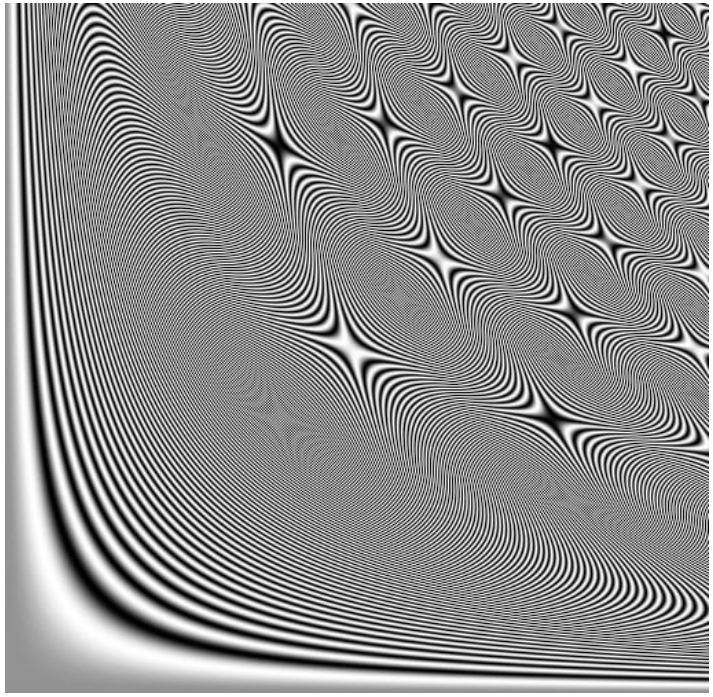
Antialiasing

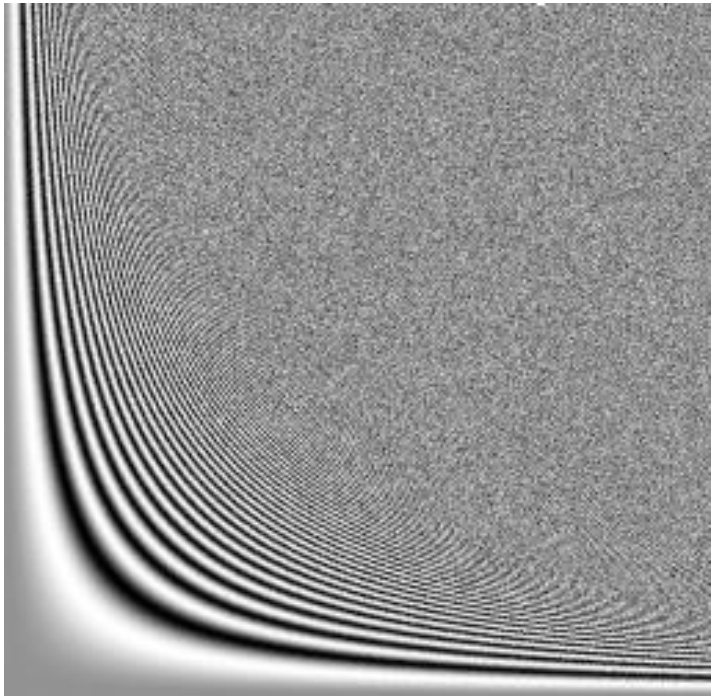
- in texturing, textures are filtered according to the given sampling rate
  - prefiltering
- in ray tracing, the sampling rate and sampling patterns are adapted
  - nonuniform sampling: tends to turn regular aliasing patterns into noise
  - adaptive sampling: use more samples in case of large variations between adjacent samples (might still miss high frequencies, small details)
- in ray tracing, radiance at a pixel position is commonly reconstructed from samples within the pixel area and samples in adjacent pixels

Antialiasing

- $f(x, y) = \frac{1}{2}(1 + \sin(x^2 y^2))$



regular sampling with aliasing



non-uniform, random sampling with noise

[Suffern]

Summary

- sampling of the continuous radiance function can cause aliasing
  - Moire patterns
  - jaggies
  - missing details
- Fourier analysis helps to understand sampling, filtering / reconstruction, and aliasing effects
- Fourier transform converts between spatial and frequency domain
- Fourier transform of the product of two functions is equivalent to the convolution of the individual Fourier transforms