Advanced Computer Graphics

Aliasing

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Outline

- motivation
- Fourier analysis
- filtering
- sampling
- reconstruction / aliasing
- antialiasing
Motivation

- incident radiance is a continuous function over the image plane
- ray tracing samples this function at discrete positions
- discrete samples
  - only approximately represent / reconstruct the original function
  - can introduce artifacts (aliasing), e.g. missing details, staircase artifacts / jaggies, erroneous patterns (Moire effect)
- goal
  - choose appropriate samples to represent the original function as good as possible
Motivation

- sampling and reconstruction
- inappropriate sampling can cause artifacts in reconstructed functions

[Reference: Foley, van Dam, Feiner, Hughes]
Motivation

- aliasing artifacts, e.g. Moire pattern

original signal  sampled signal  reconstructed signal

red - original signal
black dots - samples
blue - reconstructed signal

[Wikipedia: Alias-Effekt]
Motivation

- aliasing artifacts, e.g. jaggies
Motivation

- aliasing artifacts, e.g. missing details

[Aliasing artifacts diagram]

[Foley, van Dam, Feiner, Huges]
Motivation

- antialiasing
  - reduction of erroneous patterns
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Spectrum of a Function

- Fourier transform
  - decomposes a function into weighted sum of shifted sinusoids
  - computes amplitude and phase shift of frequencies contained in the function
  - transforms from the spatial domain to the frequency domain

\[
\mathcal{F}\{f(x)\} = F(\omega) \\
\mathcal{F}^{-1}\{F(\omega)\} = f(x)
\]

[Foley, van Dam, Feiner, Hughes]
Spectrum of a Function - Motivation

- Analysis in the frequency domain allows to understand aliasing.
- Aliasing is then reduced by:
  - Adapting the sampling.
  - Filtering the original signal (for textures).

[Wikipedia: Nyquist-Shannon-Abtasttheorem]
Spectrum of a Function - Motivation

- sampling and reconstruction can be analyzed in the frequency domain
- a band-limited function with $F(\omega) = 0$ for all $\omega > \omega_0$ has to be sampled with a frequency $\omega_{\text{sampling}} > 2\omega_0$ in order to be able to reconstruct the original function from the samples (Nyquist-Shannon sampling theorem)
- Nyquist frequency $\omega_0$
- Nyquist rate $\omega_{\text{sampling}}$
Fourier Transform - Examples

- signals in spatial and frequency domain

[Foley, van Dam, Feiner, Hughes]
Fourier Transform - Properties

- Fourier transform of the product of two functions is equivalent to the convolution of the individual Fourier transforms
  \[ \mathcal{F}\{f(x)g(x)\} = F(\omega) \otimes G(\omega) \]
- Convolution in the spatial domain is equivalent to multiplication in the frequency domain
  \[ \mathcal{F}\{f(x) \otimes g(x)\} = F(\omega)G(\omega) \]
- Important in understanding how filtering and reconstruction affect the spectrum of a function
Convolution

- \((f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau)\,d\tau\)

- convolution computes a weighted average of \(f\) using the weighting kernel \(g\)

[Wikipedia: Convolution]
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Low-Pass Filtering

- Convolution is used to filter and reconstruct functions

\[ f(x) = \frac{\sin(x)}{x} \]

Fig. 14.23 Low-pass filtering in the spatial domain. (a) Original signal. (b) Sinc filter. (c) Signal with filter, with value of filtered signal shown as a black dot (●) at filter’s origin. (d) Filtered signal. (Courtesy of George Wolberg, Columbia University.)

[Forley, van Dam, Feiner, Hughes]
Low-Pass Filtering

- sinc function in spatial and frequency domain

![sinc function](image)

.ogg

![box function](image)

[Foley, van Dam, Feiner, Huges]
Low-Pass Filtering

- Convolution with sinc function in the spatial domain corresponds to multiplication with a box function in the frequency domain.
- Given a sampling rate, this low-pass filter completely suppresses all frequency components above the Nyquist frequency.
  - Aliasing is avoided in the reconstruction process.
- Applied in texturing.
- In ray tracing, the original function cannot be simply filtered.

[Foley, van Dam, Feiner, Hughes]
Approximate Low-Pass Filtering

- truncated sinc
  - Gibbs phenomenon
- triangle
- Gaussian

[Foley, van Dam, Feiner, Hughes]
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**Sampling**

- Sampling a function corresponds to multiplying it in the spatial domain by a Dirac comb function.

\[ \mathcal{F}_T(x) = \sum_{i=-\infty}^{\infty} \delta(x - iT) \]

- [Foley, van Dam, Feiner, Hughes]
Sampling

- In frequency domain, sampling is a convolution of the function's spectrum with a Dirac comb function

\[ \mathcal{R}_T(x) \]

Dirac comb in the spatial domain

\[ \mathcal{R}_{1/T}(\omega) \]

Dirac comb in the frequency domain

\[ F(\omega) \]

Spectrum of the function

\[ |F(\omega)| \]

Spectrum of the sampled function

\[ F(\omega) \otimes \mathcal{R}_{1/T}(\omega) \]

[Foley, van Dam, Feiner, Huges]
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Reconstruction

- as a result of sampling, $F$ contains an infinite number of replications of the spectrum of the original function $f$ at multiples of the sampling frequency.
- Reconstruction tries to remove all but the spectrum of the original function by multiplying with a box filter in the frequency domain (corresponding to a convolution of the sampled function with sinc in the spatial domain).
- Our visual system reconstructs a function from pixel values.
- In ray tracing, incident radiance at pixels can be reconstructed from several samples.
Reconstruction

- sampling and reconstruction in spatial and frequency domain

[Foley, van Dam, Feiner, Hughes]
Aliasing

- if the sampling frequency is too low, the replicated copies of the spectra overlap and the spectrum of the original function cannot be reconstructed

\[ \text{Foley, van Dam, Feiner, Hughes} \]
Aliasing

spectrum of a band-limited function

spectrum of sufficiently sampled band-limited function, a box filter can reconstruct the original spectrum

spectrum of an insufficiently sampled band-limited function, the original spectrum cannot be reconstructed

[Wikipedia: Nyquist-Shannon-Abtasttheorem]
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Antialiasing

- in texturing, textures are filtered according to the given sampling rate
  - prefILTERING
- in ray tracing, the sampling rate and sampling patterns are adapted
  - nonuniform sampling: tends to turn regular aliasing patterns into noise
  - adaptive sampling: use more samples in case of large variations between adjacent samples (might still miss high frequencies, small details)
- in ray tracing, radiance at a pixel position is commonly reconstructed from samples within the pixel area and samples in adjacent pixels
Antialiasing

- \[ f(x, y) = \frac{1}{2}(1 + \sin(x^2 y^2)) \]

regular sampling with aliasing

non-uniform, random sampling with noise

[Suffern]
Summary

- sampling of the continuous radiance function can cause aliasing
  - Moire patterns
  - jaggies
  - missing details
- Fourier analysis helps to understand sampling, filtering / reconstruction, and aliasing effects
- Fourier transform converts between spatial and frequency domain
- Fourier transform of the product of two functions is equivalent to the convolution of the individual Fourier transforms