Advanced Computer Graphics

Sampling

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Outline

- introduction
- sampling strategies
- low-discrepancy sequences
- mapping samples to a disk, sphere, hemisphere
- reconstruction
- camera effects
Pixel

- radiance at a pixel is reconstructed from discrete point samples of the continuous image function
- there is no area associated with a point sample
- pixel positions are represented with integer values \((x, y)\)
- point samples in the range of \([x-0.5, x+0.5)\) and \([y-0.5, y+0.5)\) contribute to the pixel at position \((x, y)\)
- in implementations, e.g. PBRT, offsets are commonly used, i.e.
  - mapping from continuous \(c\) to integer \(d\): \(d = \lfloor c \rfloor\)
  - mapping from integer \(d\) to continuous \(c\): \(c = d + 0.5\)
  - i.e. \([x, x+1)\) and \([y, y+1)\) contribute to the pixel at position \((x, y)\)
**Introductory Example**

- **regular sampling**
  - subdivide the pixel area into a regular grid
  - trace a ray per grid cell

- **box filter**
  - compute the average incident radiance

- **effect**
  - reduced aliasing due to a higher sampling rate
  - computationally expensive

- **goal**
  - efficient sampling patterns and filter with reduced aliasing
Good Sampling Characteristics

- uniform distribution over the area
- uniform distribution of the projections in x- and y-direction
- maximal minimum distance between samples
  - avoids over- and undersampling of partial areas
- but:
- no regular spacing in x- and / or y-direction
- minimal number of samples with acceptable noise
- number of required samples depends on the application
  - sampling of a pixel area
  - sampling of time for motion blur
  - sampling of the lens area for depth of field
  - sampling of a solid angle for glossy reflection
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Random Sampling

- replaces aliasing with noise
- requires less samples compared to regular sampling
- non-uniform sampling of partial areas and projections
Random Sampling

regular sampling

25 random samples
**Stratified Sampling**

- Pixel area is subdivided into n x n strata
- One sample per stratum
- Stratified (jittered) sampling reduces clustering of samples, non-uniform sampling of areas and missing of small details
Stratified Sampling

regular sampling

64 stratified jittered samples

[Suffern]
Stratified Sampling in Higher Dimensions

- potentially generates $n^d$ samples with $d$ being the number of degrees of freedom for stratification
  - pixel area $\Rightarrow d = 2$
  - pixel area + time $\Rightarrow d = 3$
  - pixel area + time + lens area $\Rightarrow d = 5$
- instead of generating $n^5$ samples, only $n^2 + n + n^2$ samples are generated and randomly combined
- similarly, indices in different sample sets can be randomly shuffled
  - to avoid that the same sample combinations are used for different pixels
Half-jittered Sampling

- stratified (jittered) sampling can generate up to four 2D samples at the same position
- solution: choose samples closer to the center of each stratum
  - e.g.
  - **for** i=0 to \( n_x - 1 \)
    - **for** j=0 to \( n_y - 1 \)
      - \( k = i \cdot n_x - 1 \)
      - \( x_k = \text{randfrom}((i+0.25)/n_x, (i+0.75)/n_x) \)
      - \( y_k = \text{randfrom}((j+0.25)/n_y, (j+0.75)/n_y) \)
n-Rooks / Latin Hypercube Sampling (LHS)

- generate one jittered sample per row and column
- randomly shuffle the samples in x- or y-direction
- uniform distribution of the projections in x- and y-direction
- can generate an arbitrary number of stratified samples
Multi-jittered Sampling

- initial distribution in a sub grid according to n-rooks
- shuffling in x- and y-direction
- improved distribution over the area
- uniform distribution of the projections in x- and y-direction
Poisson Disk Sampling

- generate a sequence of random samples
- reject a sample, if it is too close to an existing sample
- rather expensive to compute (dart-throwing)

\[ i = 0 \]

while \( i < N \)

\[ x_i = \text{randfrom} \ [0,1) \]
\[ y_i = \text{randfrom} \ [0,1) \]

reject = false

for \( j = 0 \) to \( i-1 \)

if \( (x_i - x_j)^2 + (y_i - y_j)^2 < d^2 \)

reject = true

break

if not reject

\[ i = i+1 \]
Best-Candidate Sampling

- generate a larger number of random candidate samples within the entire sampling area
- choose the candidate farthest to previously computed samples

[Pharr, Humphreys]
Adaptive Sampling

- generate additional samples per pixel
  - if rays hit more than one shape
  - if radiance values of samples differ significantly

rendered image

adaptive sampling if more than one object intersects the rays of a pixel

adaptive sampling if the radiance per pixel varies significantly

[Pharr, Humphreys]
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Low-Discrepancy Sequences

- Good sample sets are characterized by low discrepancy
  - A given fraction of the sampling region, e.g. $[0,1]^d$, should contain the same fraction of sample points
  - Difference between the actual region and the region represented by the samples is referred to as discrepancy

$$B = \{[0,v_1] \times [0,v_2] \times \ldots \times [0,v_n]\} \quad 0 \leq v_i \leq 1$$

$$P = \{x_1, x_2, \ldots, x_n\}$$

$$D_N^*(B, P) = \sup_{b \in B} \left| \frac{\#\{x_i \in b\}}{N} - \lambda(b) \right|$$

- Low-discrepancy sequence of samples $\Rightarrow$ samples are uniformly distributed
Hammersley Sampling

- non-negative integers $k$ can be represented as
  
  $$k = a_0 + a_1p + a_2p^2 + \ldots + a_rp^r$$
  
  with $p$ being a prime and integers $a_i \in [0, p - 1]$

- $\Phi_p(k) = \frac{a_0}{p} + \frac{a_1}{p^2} + \ldots + \frac{a_r}{p^{r+1}}$ radical inverse function

- $\Phi_2(k)$ for $k = 0, 1, 2, \ldots$ is a Van der Corput sequence

- for primes $p_1, \ldots, p_{d-1}$, the $k$-th $d$-dimensional Hammersley point of a set with $n$ points is
  
  $$\left( \frac{k}{n}, \Phi_{p_1}(k), \Phi_{p_2}(k), \ldots \Phi_{p_{d-1}}(k) \right)$$
  
  $k = 0, 1, 2, \ldots, n - 1$

  $$p_1 < p_2 < \ldots < p_{d-1}$$
## Hammersley Sequence, p=2

<table>
<thead>
<tr>
<th>k</th>
<th>binary</th>
<th>binary radical inverse</th>
<th>radical inverse</th>
<th>$\phi_2(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>.1</td>
<td>1/2</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>.01</td>
<td>1/4</td>
<td>0.25</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>.11</td>
<td>1/2+1/4</td>
<td>0.75</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>.001</td>
<td>1/8</td>
<td>0.125</td>
</tr>
<tr>
<td>5</td>
<td>101</td>
<td>.101</td>
<td>1/2+1/8</td>
<td>0.635</td>
</tr>
<tr>
<td>6</td>
<td>110</td>
<td>.011</td>
<td>1/4+1/8</td>
<td>0.325</td>
</tr>
<tr>
<td>7</td>
<td>111</td>
<td>.111</td>
<td>1/2+1/4+1/8</td>
<td>0.875</td>
</tr>
<tr>
<td>8</td>
<td>1000</td>
<td>.0001</td>
<td>1/16</td>
<td>0.0625</td>
</tr>
</tbody>
</table>

$$D_N^* = O\left(\frac{\log N}{N}\right)$$
Hammersley Sampling in 2D

- \( \left( \frac{k}{n}, \Phi_2(k) \right) \quad k = 0, 1, 2, \ldots, n - 1 \)

\[
\begin{align*}
p' &= p, \quad k' = k, \quad \phi = 0 \\
\text{while } k' > 0 \text{ do} \\
& \quad a = k' \mod p \\
& \quad \phi = \phi + a/p' \\
& \quad k' = \text{int} \left( k'/p \right) \\
& \quad p' = p' \times p
\end{align*}
\]

16 samples

256 samples

computation of the k-th Hammersley value with basis p

[Suffern]
Halton Sampling in 2D

- allows to successively generate additional samples (in contrast to Hammersley)
- \( D_N^* = O \left( \frac{(\log N)^d}{N} \right) \)
- \((\Phi_{p_1}(k), \Phi_{p_2}(k))\) \(k = 0, 1, 2, \ldots\)
Hammersley vs. Halton - 2D Area

Wong, Luk, Heng

Hammersley sampling

Halton sampling

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Sphere - Hammersley vs. Halton

\[(\frac{k}{n}, \Phi_p(k)) \rightarrow (\phi, t) \in [0, 2\pi) \times [-1, 1] \rightarrow (\sqrt{1 - t^2} \cos \phi, \sqrt{1 - t^2} \sin \phi, t)\]
Disk - Rejection Sampling

- sample points outside a disk are rejected
- no distortion
- but differing number of remaining samples
Disk - Concentric Map

- mapping from square to disk
- minimal distortion
- number of samples is preserved

- e.g., \((x > y, x > -y) \rightarrow (r = x, \phi = \frac{\pi}{4} \frac{y}{x})\)
- four quarters

[Suffern]
Hemisphere

- mapping from square to hemisphere with a cosine power density distribution
- surface density of samples varies with $\theta$ according to $d = \cos^m \theta$
**Hemisphere**

- **mapping:**
  
  $$(x \in [0, 1), y \in [0, 1)) \rightarrow (\phi = 2\pi x, \theta = \cos^{-1}\left[(1 - y)^{1/(m+1)}\right])$$

Local orthonormal basis at the center of the hemisphere:

$$\mathbf{p} = \sin \theta \cos \phi \mathbf{u} + \sin \theta \sin \phi \mathbf{v} + \cos \theta \mathbf{w}$$
Hemisphere Sampling

\[ m = 1000 \quad m = 100 \quad m = 10 \]

\[ m = 1 \quad m = 1 \quad m = 0 \]
Applications

- area sampling is used to sample pixel areas
- hemisphere sampling is used for global illumination effects
- disk sampling is used for the depth-of-field effect
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Filtering Principle

- pixel values are reconstructed from the radiance values of adjacent samples

$$I(x, y) = \frac{\sum_i f(x-x_i, y-y_i) L(x_i, y_i)}{\sum_i f(x-x_i, y-y_i)}$$

- $f$ is a filter function that weights the influence of a sample $(x_i, y_i)$ according to its distance to $(x, y)$

The extent can be larger than the pixel size.
Box- / Triangle Filter

- computationally efficient
- bad reconstruction characteristics

Box filter applied to a step function
Box filter applied to a sinusoidal function with increasing frequency ⇒ introduces post-aliasing

[Pharr, Humphreys]
Gaussian-/Mitchell Filter

- reasonably good reconstruction
- introduces blurring
- e.g., 1D filter

\[ f(x) = e^{-\alpha x^2} - e^{-\alpha w^2} \]

Mitchell filter applied to a step function (minimal ringing)

Mitchell filter applied to a sinusoidal function with increasing frequency ⇒ more aliasing compared to Mitchell due to undersampling

Gaussian filter

Mitchell filter

[Pharr, Humphreys]
Truncated Sinc Filter

- reasonably good reconstruction
- introduces blurring

Sinc filter applied to a step function (some ringing)

Sinc filter applied to a sinusoidal function with increasing frequency \(\Rightarrow\) aliasing due to undersampling

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[Pharr, Humphreys]

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Summary

- sampling strategies for pixel area, disk, hemisphere help to reduce aliasing or to replace it with noise
  - stratified sampling
  - Poisson disk sampling
  - low discrepancy sequences
  - mapping to disk, hemisphere, sphere
- reconstruction of pixel values from sample radiances
  - box
  - Mitchell
  - truncated sinc
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Multidimensional Sampling

- sampling the pixel area
  - captures the continuous radiance function
- sampling a time period
  - captures motion blur effects
- sampling a disk
  - captures depth-of-field effects

[Shirley, Morley]
Thin Lens

- instead of a pinhole camera, we model a thin lens
  - every point \( p \) on the focal plane has a corresponding point \( p' \) on the image plane
  - every ray that goes through \( p \) and the lens also goes through \( p' \)
  - a ray through the center of the lens is not refracted

![Diagram of thin lens](image.png)
Circle of Confusion

- $q$ is not on the focal plane
- rays that go through $q$ and the lens do not intersect at a point on the image plane
- instead, the intersections with the image plane form the circle of confusion
Depth-of-Field

- is the range of distances to the lens in which the scene is in focus, i.e. the circle of confusion is smaller than the area of a pixel
- in cameras, the aperture is used to adapt the depth-of-field
  - narrow aperture $\Rightarrow$ large range of distances that are in focus
  - wide aperture $\Rightarrow$ small range of distances that are in focus
**Simplified Model**

- lens is modeled as disk placed at the eye position
- focal plane is defined by the user
- \( p \) is computed using a center ray from the center of a disk through a sample point in the view plane
- rays from the sampled disk through \( p \) are generated
- if these rays hit an object on the focal plane, the object is perfectly reconstructed

[Suffern]
Simplified Model

- center ray
  - computes \( p \)
  - does not return a radiance

- primary rays
  - start at a sample of the disk
  - into the direction of \( p \)
  - return a radiance that is associated with the pixel sample that has been used for the center ray

- size of the disk governs the blurring effect

For a real lens, the black rays would be refracted. The rays would intersect at a point on the real image plane behind the lens.
Implementation

- in view / camera space
  - a ray is computed for a lens sample $l_s$ into the direction $d = p - l_s$

\[
\begin{align*}
  p &= (p_{sx} \frac{f}{d}, p_{sy} \frac{f}{d}, -f) \\
  d &= (p_x - l_{sx}, p_y - l_{sy}, -f)
\end{align*}
\]

- ray equation
  \[
  r(t) = l_s + t \frac{d}{\|d\|}
  \]
Sampling

- sampling of the pixel area
- either
  - generate a center ray per pixel sample
  - sample the disk per center ray
- or
  - sample the disk
  - associate one disk sample with one pixel sample
  - i.e., use different center rays for all disk samples
Results

- 100 random pixel and disk samples
- one-to-one mapping of pixel and disk samples
- varying size of the disk