Advanced Computer Graphics
Stochastic Raytracing 1

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From Radiosity to Raytracing

- Radiosity equation governs light transport for diffuse surfaces. ⇒ How to describe light transport for general surfaces?
- How to solve for the light transport?
- How to compute the relevant part of the light transport towards a sensor?
Stochastic Raytracing

- Light transport towards the sensor requires to solve
  \[ L(p \rightarrow \omega_o) = L_e(p \rightarrow \omega_o) + \int_S f_r(p, \omega_i \leftrightarrow \omega_o) L(p' \rightarrow -\omega_i) G(p, p') dA_{p'} \]
- Monte Carlo integration approximates the reflectance integral
  - E.g., \( \sum_i f_r(p, \omega_i \leftrightarrow \omega_o) L(p' \rightarrow -\omega_i) G(p, p') A_{p'} \)
  - Trace rays into the scene
  - Compute radiance along this ray
  - Associate an area / solid angle with each ray
  - Accumulate all contributions
Outline

- Diffuse vs. general global illumination
- Monte Carlo integration
- Sampling of random variables
Governing Equations

- Rendering equation
  - Governing equation for general global illumination methods
    \[ L(p \rightarrow \omega_o) = L_e(p \rightarrow \omega_o) + \int_S f_r(p, \omega_i \leftrightarrow \omega_o) L(x \rightarrow -\omega_i) V(p, x) \frac{\cos(\omega_i, n_p) \cos(-\omega_i, n_x)}{r_{px}^2} \, dA_x \]

- Radiosity equation
  - Governing equation for diffuse global illumination methods
    \[ L(p \rightarrow \omega_o) = \frac{B(p)}{\pi} \quad f_r(p, \omega_i \leftrightarrow \omega_o) = \frac{\rho(p)}{\pi} \]
    \[ B(p) = B_e(p) + \frac{\rho(p)}{\pi} \int_S B(x) V(p, x) \frac{\cos(\omega_i, n_p) \cos(-\omega_i, n_x)}{r_{px}^2} \, dA_x \]
A Solution Strategy - Radiosity

- Finite Element Method
- Start with a continuous form / function
  \[ B(p) = B_e(p) + \frac{\rho(p)}{\pi} \int_S B(x)V(p, x) \frac{\cos(\omega_i, n_p) \cos(-\omega_i, n_x)}{r_{px}^2} dA_x \]
- Discretization
  \[ B = B_e + FB \]
  \[ B = (I - F)^{-1} B_e \]
- Solving for a vector with unknown radiosities
  \[ (I - F)^{-1} = \sum_{k=0}^{\infty} F^k \]
  \[ B = B_e + FB_e + FFB_e + FFFB_e + \ldots \]
An Alternative Strategy

- Start with the general form of the rendering equation, e.g. in hemispherical form

\[
L(p \rightarrow \omega_o) = L_e(p \rightarrow \omega_o) + \int_\Omega f_r(p, \omega_i \leftrightarrow \omega_o) L(p \leftrightarrow \omega_i) \cos(\omega_i, n_p) d\omega_i
\]

- Solving for a function of unknown radiances \( L(p \rightarrow \omega_o) \)
  - I.e., radiance at all surface positions into all directions
Operator Form of the Rendering Equation

- Operators transform a function into another one
- Scattering operator

\[(Kh)(p \rightarrow \omega_o) = \int_{\Omega} f_r(p, \omega_i \leftrightarrow \omega_o) h(p \leftarrow \omega_i) \cos(\omega_i, n_p) d\omega_i\]

- Applied to an incident radiance function \(L(p \leftarrow \omega_i)\), exitant radiance after one bounce / scattering step is returned

\[L(p \rightarrow \omega_o) = (KL)(p \leftarrow \omega_i)\]

- \(K\) operates on an entire function, i.e. on all incident radiances for all positions \(p\) and direction \(\omega_i\)

Operator Form of the Rendering Equation

- Propagation operator

\[(Gh)(p \leftarrow \omega_i) = h(p' \rightarrow -\omega_i)\]  
\(p'\) indicates the raycast operator applied to \(p\)

- Applied to an exitant radiance function \(L(p' \rightarrow -\omega_i)\), incident radiance at \(p\) from direction \(\omega_i\) is returned

\[L(p \leftarrow \omega_i) = (GL)(p' \rightarrow -\omega_i)\]

- Radiance is preserved / propagated along the line between \(p\) and \(p'\)

- \(p\) and \(p'\) can be reversed, i.e. \(L(p' \leftarrow -\omega_o) = (GL)(p \rightarrow \omega_o)\)

Operator Form of the Rendering Equation

– Light transport operator

\[ T = KG \]

– Composition of scattering and propagation

– Maps an exitant radiance function to the exitant radiance function after one scattering step

– Remember: \( G \) maps exitant radiance to incident radiance propagated along a direction. Then, \( K \) maps incident radiance to exitant radiance after scattering

Operator Form of the Rendering Equation

\[ L(p \to \omega_o) = L_e(p \to \omega_o) + \int_{\Omega} f_r(p, \omega_i \leftrightarrow \omega_o) L(p \leftarrow \omega_i) \cos(\omega_i, n_p) \, d\omega_i \]

- Can be written as

\[ L(p \to \omega_o) = L_e(p \to \omega_o) + (KGL)(p \to \omega_o) \]

- Light transport equation

\[ L = L_e + TL \] Infinite number of equations with an infinite number of unknown exitant radiances
- \( T \) relates exitant radiance functions
- Represents the light propagation equilibrium
Light Transport Equation

\[ L = L_e + TL \]

– Solving for the unknown radiance function

\[(I - T)L = L_e\]
\[L = (I - T)^{-1}L_e\]

– Neumann series

\[ L = \sum_{k=0}^{\infty} (T^k L_e) \]
\[ \approx L_e + TL_e + TTL_e + TTTL_e + \ldots \]
Light Transport Equation

- Discussion
  - Radiance function is a sum of
    - Emitted radiance \( L_e \)
    - Emitted radiance after one scattering \( TL_e \)
    - Emitted radiance after two scatterings \( T^2 L_e \)
    - ...

\[ L \approx L_e + TL_e + T^2 L_e + T^3 L_e + \ldots \]
Terms in the Neumann Series

- Example contributions to terms
Forward Raytracing

- Send rays / propagate radiance from all light source positions into all directions $\Rightarrow L_e$

- At all intersection points $p$, solve the integral
  \[ L_1(p \rightarrow \omega_o) = \int_{\Omega} f_r(p, \omega_i \leftrightarrow \omega_o) G L_e \cos(\omega_i, n_p) d\omega_i \]
  for all direction $\omega_o \Rightarrow T L_e$

- Trace rays to propagate $T L_e$

- At all intersection points $p$, solve the integral
  \[ L_2(p \rightarrow \omega_o) = \int_{\Omega} f_r(p, \omega_i \leftrightarrow \omega_o) G L_1 \cos(\omega_i, n_p) d\omega_i \]
  for all direction $\omega_o \Rightarrow T T L_e$
Forward Raytracing

- At a sensor: Accumulate radiance contributions of rays after $n$ scattering steps, i.e. compute $L_e + TL_e + T^2L_e + \ldots$
Operator Form of the Rendering Equation

\[ L(p \to \omega_o) = L_e(p \to \omega_o) + \int_\Omega f_r(p, \omega_i \leftrightarrow \omega_o) L(p \leftrightarrow \omega_i) \cos(\omega_i, n_p) d\omega_i \]

– Can be written as

\[ L(p \to \omega_o) = L_e(p \to \omega_o) + (KGL)(p \to \omega_o) \]

– Light transport equation

\[ L = L_e + TL \quad \text{Infinite number of equations with an infinite number of unknown exitant radiances} \]

– \( T \) relates exitant radiance functions

– Represents the light propagation equilibrium
Light Transport Equation

\[ L = L_e + TL \]

- Solving for the unknown radiance function

\[ (I - T)L = L_e \]

\[ L = (I - T)^{-1}L_e \]

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\[ L = \sum_{k=0}^{\infty} (T^k L_e) \]

\[ \approx L_e + TL_e + T^2L_e + T^3L_e + \ldots \]
Light Transport Equation

– Discussion

– Radiance function is a sum of
  – Emitted radiance \( L_e \)
  – Emitted radiance after one scattering \( TL_e \)
  – Emitted radiance after two scatterings \( T^2 L_e \)
  – ...

\[
L \approx L_e + TL_e + T^2 L_e + T^3 L_e + \ldots
\]
Backward Raytracing

– Consider rays from the sensor into the scene
– Propagate radiance from visible light sources
– \( \Rightarrow \) part of \( L_e \) visible to the sensor
– At intersection points \( p \) with the scene, compute radiance
  \[
  L(p \rightarrow \omega_o) = \int_\Omega f_r(p, \omega_i \leftrightarrow \omega_o) L_e(p \leftarrow \omega_i) \cos(\omega_i, n_p) d\omega_i
  \]
  that is propagated in direction \( \omega_o \) towards the sensor
– \( \Rightarrow \) part of \( TL_e \) visible to the sensor
– ...
Backward Raytracing

– Trace rays from the sensor into the scene
Setting at Sensor

\[ L(p_0 \leftarrow -\omega_o) = \]
\[ L(p_1 \rightarrow \omega_o) \]

\[ \begin{align*}
\vec{p}_0 & \quad \vec{-\omega_o} \\
\vec{q} & \quad \vec{l} \\
\vec{q'} & \quad \vec{-l}
\end{align*} \]

\[ L(q \leftarrow l) = \]
\[ L_e(q' \rightarrow -l) \]

- How to compute \( L(p_1 \rightarrow \omega_o) \) and what is its relation to \( L_e + TL_e + TTTL_e + \ldots \)?
Setting at First-Level Intersections

\[ L(p_1 \rightarrow \omega_o) = \int_S f_r(p_1, \omega_i \leftrightarrow \omega_o) L(p_2 \rightarrow -\omega_i) G(p_1, p_2) dA_{p_2} \]

\[ = \int_{\text{Light Sources}} f_r(p_1, \omega_i \leftrightarrow \omega_o) L_e(p_2 \rightarrow -\omega_i) G(p_1, p_2) dA_{p_2} \]

\[ + \int_{\text{Scene}} f_r(p_1, \omega_i \leftrightarrow \omega_o) L(p_2 \rightarrow -\omega_i) G(p_1, p_2) dA_{p_2} \]

- \( \int_{\text{Light Sources}} \) \( \cdots \) is the part of \( TL_e \) visible to the sensor

- Computation of \( \int_{\text{Scene}} \) \( \cdots \) requires \( L(p_2 \rightarrow -\omega_i) \)

\[ L(p_1 \leftrightarrow \omega_i) = \]

\[ L(p_2 \rightarrow -\omega_i) \]

towards sensor

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Setting at Second-Level Intersections

\[ L(p_2 \rightarrow \omega_o) = \int_S f_r(p_2, \omega_i \leftrightarrow \omega_o) L(p_3 \rightarrow -\omega_i) G(p_2, p_3) \, dA_{p_3} \]

\[ = \int_{\text{Light Sources}} f_r(p_2, \omega_i \leftrightarrow \omega_o) L_e(p_3 \rightarrow -\omega_i) G(p_2, p_3) \, dA_{p_3} \]

\[ + \int_{\text{Scene}} f_r(p_2, \omega_i \leftrightarrow \omega_o) L(p_3 \rightarrow -\omega_i) G(p_2, p_3) \, dA_{p_3} \]

- \( \int_{\text{Light Sources}} \cdots \) is the part of \( L e \) visible to the sensor
- Computation of \( \int_{\text{Scene}} \cdots \) requires \( L(p_3 \rightarrow -\omega_i) \)

\[ L(p_2 \leftrightarrow \omega_i) = \]

\[ L(p_3 \rightarrow -\omega_i) \]
Summary

- Recursive evaluation of

\[
L(p \rightarrow \omega_o) = \int_S f_r(p, \omega_i \leftrightarrow \omega_o) L(p' \rightarrow -\omega_i) G(p, p') dA_{p'}
\]

\[
= \int_{\text{Light Sources}} f_r(p, \omega_i \leftrightarrow \omega_o) L_e(p' \rightarrow -\omega_i) G(p, p') dA_{p'}
\]

\[
+ \int_{\text{Scene}} f_r(p, \omega_i \leftrightarrow \omega_o) L(p' \rightarrow -\omega_i) G(p, p') dA_{p'}
\]

- Each recursion level computes parts of the functions \(L_e, TL_e, T^2L_e, \ldots\) that are visible to the sensor
Numerical Integration

- The integral $\int_S \ldots$ is approximately computed with a sum of samples $\sum_i \ldots$
- For each sample $i$,
  - A ray is cast into the scene
  - Intersection with the scene is computed
  - Radiance along the ray is computed
Numerical Integration

- Typically, \( \int_S \cdots = \int_{\text{Scene}} \cdots + \int_{\text{Light Sources}} \cdots \approx \sum_{\text{Scene}_i} \cdots + \sum_{\text{Light Source}_i} \cdots \) is considered.
- For \( \sum_{\text{Light Source}_i} \cdots \), light source areas are sampled and rays towards those positions are processed.
- For \( \sum_{\text{Scene}_i} \cdots \), the respective solid angle is sampled and rays towards those directions are processed.
Numerical Integration

- Due to the recursive nature, the number of processed rays grows exponentially with the recursion level.
- ⇒ Monte Carlo integration
  - Efficient for multidimensional integral
  - Adaptive sample distribution
  - Very flexible in terms of the number of used samples
  - Even one sample can be used to approximate an integral
- ⇒ e.g., Path tracing
  - At each recursion level, trace a fixed number of rays to light sources and one ray into the scene (which generates a ray path).
Advanced Computer Graphics
Stochastic Raytracing 2

Matthias Teschner
Outline

- Diffuse vs. general global illumination
- Monte Carlo integration
- Sampling of random variables
Goal

- Approximating the solution of the light transport equation
  \[ L = \sum_{k=0}^{\infty} (T^k L_e) \]
- Recursive evaluation of
  \[ L(p \rightarrow \omega_o) = L_e(p \rightarrow \omega_o) + \int_S f_r(p, \omega_i \leftrightarrow \omega_o) L(p' \rightarrow -\omega_i) G(p, p') \, dA_{p'} \]
- Each recursion level computes parts of the functions \( L_e, TL_e, T^2L_e, \ldots \) that are visible to the sensor
Numerical Integration – Fixed Sample Size

- E.g. Riemann sum
  - \[ \int_a^b f(x) \, dx \approx \sum_i f(x_i) \Delta x \quad \Delta x = \frac{b-a}{N} \]
- More / smaller samples \( \Rightarrow \) better accuracy
- \( d \) dimensional integrals require \( N^d \) samples
Numerical Integration – Adaptive Sample Size

- E.g., Monte Carlo integration
  \[ \int_a^b f(x)\,dx \approx \sum_i f(x_i)\Delta x_i, \quad \text{adaptive sample size } \Delta x_i \]
- More / smaller samples \(\Rightarrow\) better accuracy
- \(d\) dimensional integrals work with arbitrary sample numbers
- Sample size is only approximated \(\Rightarrow\) noise
Stochastic Raytracing - Concept

- Approximately evaluate the reflectance integral
  \[ \int_{\Omega} f_r(p, \omega_i \leftrightarrow \omega_o) L(p \leftarrow \omega_i) \cos(\omega_i, n_p) d\omega_i \]
  by
  - Tracing rays into \textit{randomly} sampled 2D directions
  - Computing the incoming radiances

- Integral is approximated with
  \[ \sum_i f_r(p, \omega_i \leftrightarrow \omega_o) L(p \leftarrow \omega_i) \cos(\omega_i, n_p) \Delta\Omega_i \]
  - 2 dimensional sample directions \( \omega_i = (\theta_i, \phi_i) \)
  - \( \Delta\Omega_i \) is an approximation of the solid angle of sample direction \( \omega_i = (\theta_i, \phi_i) \)
Introduction

– Challenges
  – Approximate the integral as exact as possible
  – Trace as few rays as possible / use as few samples as possible
  – Trace relevant rays / use relevant samples
    – Rays / samples to light sources are very relevant
      (Rays / samples to occluded light sources are irrelevant)
    – For diffuse surfaces, rays / samples in normal direction are more relevant than rays / samples perpendicular to the normal
    – For specular surfaces, rays / samples in reflection direction are relevant
Properties

- Benefits
  - Processes only evaluations of the integrand at arbitrary surface points in the domain
  - Works for a large variety of integrands, e.g., it handles discontinuities
  - Appropriate for integrals of arbitrary dimensions
  - Allows for non-uniform sample patterns / adaptive sample sizes
Properties

- Drawbacks
  - Using \( n \) samples, the scheme converges to the correct result with \( O \left( n^{1/2} \right) \)
  - I.e., to half the error, \( 4n \) samples are required
  - Errors are perceived as noise, i.e. pixels are arbitrarily too bright or dark (due to the erroneous approximation of the sample size)
  - Evaluation of the integrand at a point and for a direction is expensive (ray intersection tests)
Continuous Random Variables

- Motivation: random sampling of directions
- Continuous random variables $x$
  - In contrast to discrete random variables, infinite number of possible values
- Canonical uniform random variable $0 \leq \xi < 1$
  - Sample sets with arbitrary distributions can be computed from $\xi$
Probability Density Function PDF $p(x)$

- Motivation: PDF governs the size / solid angle of a sample / sample direction
- Probability of a random variable taking certain value ranges
  - $p(x) \geq 0 \quad \forall x \in [a, b]$ 
  - $P(x_0 \leq X \leq x_1) = \int_{x_0}^{x_1} p(x) \, dx$ 
  - $\int_{a}^{b} p(x) \, dx = 1$ 
  - Example
    - Uniform PDF for $0 \leq X \leq 5$
    - $1 = \int_{0}^{5} p(x) \, dx = p(x) \int_{0}^{5} \, dx = 5 \, p(x)$ 
    - $p(x) = \frac{1}{5}$
Cumulative Distribution Function CDF $P(x)$

- Motivation: CDFs are required to generate sample sets for arbitrary PDFs from uniform sample sets
- Probability of a random variable to be less or equal to $x$
  - $P(x) = Pr(X \leq x) = \int_a^x p(x)dx$
  - $P(a) = 0 \leq P(x) \leq 1 = P(b)$
  - $Pr(x_0 \leq X \leq x_1) = P(x_1) - P(x_0)$
Expected Value

- Motivation: expected value of an estimator function is equal to the reflectance integral
- Expected value $E[f(x)]$ of a function $f(x)$ is defined as the weighted average value of the function over a domain $D$
  
  $$E[f(x)] = \int_D f(x) \, p(x) \, dx \quad \text{with} \quad \int_D p(x) \, dx = 1$$

- Properties
  
  $$E[af(x)] = aE[f(x)]$$
  $$E[\sum_i f(X_i)] = \sum_i E[f(X_i)]$$

  For independent random variables $X_i$
Expected Value

- Examples for uniform PDF $p(x)$
  
  - $f(x) = \cos(x) \quad D = [0, \pi] \quad p(x) = \frac{1}{\pi}$
    
    $E[\cos(x)] = \int_0^\pi \cos(x) \frac{1}{\pi} \, dx = \frac{1}{\pi} (-\sin \pi + \sin 0) = 0$
  
  - $f(x) = x \quad D = [0, 6] \quad p(x) = \frac{1}{6}$
    
    $E[x] = \int_0^6 x \frac{1}{6} \, dx = \frac{1}{6} (\frac{6^2}{2} - 0) = 3$
  
  - $f(x)$

\[ E[f(x)] = \frac{1}{b-a} \int_a^b f(x) \, dx \]

\[ \int_a^b f(x) \, dx = E[f(x)](b - a) \]
Monte Carlo Estimator - Uniform Random Variables

- Motivation: approximation of the reflectance integral
- Goal: computation of $\int_{a}^{b} f(x) dx$
- Uniformly distributed random variables $X_i \in [a, b]$
- Probability density function $p(x) = \frac{1}{b-a}$ Constant and integration to one
- Monte Carlo estimator $F_N = \frac{b-a}{N} \sum_{i=1}^{N} f(X_i)$
- Expected value of $F_N$ is equal to the integral $\int_{a}^{b} f(x) dx$
  
  $E[F_N] = \int_{a}^{b} f(x) dx$
Monte Carlo Estimator - Uniform Random Variables

\[
E[F_N] = E\left[ \frac{b-a}{N} \sum_{i=1}^{N} f(X_i) \right]
\]

\[
= \frac{b-a}{N} \sum_{i=1}^{N} E[f(X_i)]
\]

\[
= \frac{b-a}{N} \sum_{i=1}^{N} \int_{a}^{b} f(x)p(x)dx
\]

\[
= \frac{b-a}{N} \sum_{i=1}^{N} \int_{a}^{b} f(x) \frac{1}{b-a}dx
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \int_{a}^{b} f(x)dx
\]

\[
= \int_{a}^{b} f(x)dx
\]
Monte Carlo Estimator - Uniform Random Variables

- PDF \( p(x) = \frac{1}{b-a} \)
- Estimator \( F_N = \frac{b-a}{N} \sum_{i=1}^{N} f(X_i) \)
- Integral
  - \( \int_{a}^{b} f(x)dx \approx \frac{b-a}{N} \sum_{i=1}^{N} f(X_i) = \sum_{i=1}^{N} f(X_i) \frac{b-a}{N} = \sum_{i=1}^{N} f(X_i) \frac{1}{N p(X_i)} \)
- Function value \( f(X_i) \)
- Approximate sample size \( \frac{1}{N p(X_i)} \)
Examples - Uniform Random Variables

- Integral \( \int_0^1 5x^4 \, dx = 1 \)
- Estimator \( F_N = \frac{1-0}{N} \sum_{i=1}^N 5X_i^4 \)
  Sample size approx. 1/N
- For an increasing number of uniformly distributed random variables \( X_i \), the estimator converges to one

\[
F_N = \sum_{i=1}^N f(X_i) \frac{b-a}{N} \\
F_N = (b-a) \frac{1}{N} \sum_{i=1}^N f(X_i) \\
= (b-a)f(x) \\
E[F_N] = \int_a^b f(x) \, dx
\]
Monte Carlo Estimator - Non-uniform Random Variables

- Monte Carlo estimator

\[ F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)} \quad p(X_i) \neq 0 \]

- \( E[F_N] = E \left[ \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)} \right] \)

\[ = \frac{1}{N} \sum_{i=1}^{N} \int_{a}^{b} \frac{f(x)}{p(x)} p(x) \, dx \]

\[ = \frac{1}{N} \sum_{i=1}^{N} \int_{a}^{b} f(x) \, dx \]

\[ = \int_{a}^{b} f(x) \, dx \]
Monte Carlo Estimator - Non-uniform Random Variables

PDF \( p(x) \)

Estimator \( F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)} \)

Integral

\[
\int_{a}^{b} f(x)dx \approx \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)} = \sum_{i=1}^{N} f(X_i) \frac{1}{N \ p(X_i)}
\]

Function value \( f(X_i) \)

Approximate sample size \( \frac{1}{N \ p(X_i)} \)
Approximate Sample Size

- Sample size / distance for uniform PDF: \( \approx \frac{b-a}{N} = \frac{1}{Np(X_i)} \)

- Sample size for non-uniform PDF: \( \approx \frac{1}{Np(X_i)} \)
Monte Carlo Estimator - Multiple Dimensions

- E.g., \( \int_{x_0}^{x_1} \int_{y_0}^{y_1} f(x, y) \, dx \, dy \)
- Samples \( x_i \) are two-dimensional
- Uniformly distributed random samples
  \( (x_0, y_0) \leq x_i = (x_i, y_i) \leq (x_1, y_1) \)
- Probability density function
  \( p(x_i) = \frac{1}{x_1 - x_0} \frac{1}{y_1 - y_0} \)
- Monte Carlo estimator
  \( F_N = \frac{(x_1 - x_0)(y_1 - y_0)}{N} \sum_{i=1}^{N} f(x_i) \)
- Approximate sample size is
  \( \frac{(x_1 - x_0)(y_1 - y_0)}{N} \)
- E.g., \( \int_1^4 \int_1^4 f(x, y) \, dx \, dy \)
- Uniformly distributed random samples
- Probability density function \( p(X_i) = \frac{1}{4-1} \frac{1}{4-1} = \frac{1}{9} \)
- Monte Carlo estimator \( F_N = \frac{9}{N} \sum_{i=1}^{N} f(X_i) \)
- Approximate sample size \( \frac{9}{N} \)
Monte Carlo Estimator - Integration over a Hemisphere

- Approximate computation of the irradiance at a point

\[ E_i(p) = \int_{2\pi} L_i(p, \omega) \cos \theta d\omega \]

\[ = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} L_i(p, \theta, \phi) \cos \theta \sin \theta d\theta d\phi \]

- Estimator

\[ F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)} = \frac{1}{N} \sum_{i=1}^{N} \frac{L_i(p,\theta_i,\phi_i) \cos \theta_i \sin \theta_i}{p(\theta_i,\phi_i)} \]

- Choosing a PDF

- Should be similar to the shape of the integrand
- As incident radiance is weighted with \( \cos \theta \), it is appropriate to generate more samples close to the top of the hemisphere
- \( p(\theta, \phi) \propto \cos \theta \)

This flexibility is an important aspect of Monte Carlo integration.
Monte Carlo Estimator - Integration over a Hemisphere

- Probability distribution
  \[
  \int_{2\pi} c \, \tilde{p}(\omega) d\omega = 1 \quad \tilde{p}(\theta, \phi) = \cos \theta \\
  \int_0^{2\pi} \int_0^{\frac{\pi}{2}} c \, \cos \theta \sin \theta \, d\theta \, d\phi = 1 \\
  c \cdot \frac{2\pi}{1+1} = 1 \\
  c = \frac{1}{\pi} \\
  p(\theta, \phi) = \frac{\cos \theta \sin \theta}{\pi}
  \]

- Estimator
  \[
  F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{L_i(p, \theta_i, \phi_i) \cos \theta_i \sin \theta_i}{p(\theta_i, \phi_i)} \\
  = \frac{\pi}{N} \sum_{i=1}^{N} L_i(p, \theta_i, \phi_i) \approx \int_0^{2\pi} \int_0^{\frac{\pi}{2}} L_i(p, \theta, \phi) \cos \theta \sin \theta \, d\theta \, d\phi
  \]
  If \( \theta \) and \( \phi \) are sampled according to PDF \( p(\theta, \phi) \)
Monte Carlo Estimator - Integration over a Hemisphere

- Integral \( \int_0^{2\pi} \int_0^\frac{\pi}{2} L_i(p, \theta, \phi) \cos \theta \sin \theta d\theta d\phi \)
- PDF \( p(\theta, \phi) = \frac{\cos \theta \sin \theta}{\pi} \)
- Estimator \( \frac{\pi}{N} \sum_{i=1}^N L_i(p, \theta_i, \phi_i) \)
  \[= \sum_{i=1}^N L_i(p, \theta_i, \phi_i) \cos \theta_i \frac{\pi}{N \cos \theta_i} \]
- Function value \( L_i(p, \theta_i, \phi_i) \cos \theta_i \) for direction \((\theta_i, \phi_i)\)
- Approximate sample size / solid angle \( \frac{\pi}{N \cos \theta_i} \)
  - For large \( N \)
- The PDF in terms of the solid angle is \( p(\omega_i) = \frac{\cos \theta_i}{\pi} \)
Monte Carlo Integration - Steps

- Choose an appropriate probability density function
- Generate random samples according to the PDF
- Evaluate the function for all samples
- Accumulate sample values weighted with their approximate sample size
Monte Carlo Estimator - Error Reduction

- Importance sampling
  - Motivation: contributions of larger sample values are more important
  - PDF should be similar to the shape of the function
  - Optimal PDF \( p(x) = \frac{f(x)}{\int f(x) \, dx} \)
  - E.g., if incident radiance is weighted with \( \cos \theta \), the PDF should choose more samples close to the normal direction

[Suffern]
Monte Carlo Estimator - Error Reduction

- Stratified sampling
  - Domain subdivision into strata
  - E.g., handling direct and indirect illumination differently

\[
L(p \to \omega_o) = L_e(p \to \omega_o) + \int_S f_r(p, \omega_i \leftrightarrow \omega_o)L(p' \to -\omega_i)G(p, p')dA_{p'}
\]

\[
= L_e(p \to \omega_o) + \int_{\text{Light Sources}} f_r(p, \omega_i \leftrightarrow \omega_o)L_e(p' \to -\omega_i)G(p, p')dA_{p'}
\]

\[
+ \int_{\text{Scene}} f_r(p, \omega_i \leftrightarrow \omega_o)L(p' \to -\omega_i)G(p, p')dA_{p'}
\]
Advanced Computer Graphics
Stochastic Raytracing 3

Matthias Teschner
Monte Carlo Estimator - Integration over a Hemisphere

- Approximate computation of the irradiance at a point
  \[ E_i(p) = \int_{2\pi}^{2\pi} L_i(p, \omega) \cos \theta d\omega \]
  \[ = \int_{0}^{2\pi} \int_{0}^{\frac{\pi}{2}} L_i(p, \theta, \phi) \cos \theta \sin \theta d\theta d\phi \]

- Estimator
  \[ F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)} = \frac{1}{N} \sum_{i=1}^{N} \frac{L_i(p, \theta_i, \phi_i) \cos \theta_i \sin \theta_i}{p(\theta_i, \phi_i)} \]

- Choosing a PDF
  - This flexibility is an important aspect of Monte Carlo integration.
  - Should be similar to the shape of the integrand
  - As incident radiance is weighted with \( \cos \theta \), it is appropriate to generate more samples close to the top of the hemisphere
  - \( p(\theta, \phi) \propto \cos \theta \)
Outline

− Diffuse vs. general global illumination
− Monte Carlo integration
− Sampling of random variables
  − Inversion method
  − Rejection method
  − Transforming between distributions
  − 2D sampling
  − Examples
Motivation - Rendering Equation

- Hemispherical form

\[ L(p \rightarrow \omega_o) = L_e(p \rightarrow \omega_o) + \int_{\Omega} f_r(p, \omega_i \leftrightarrow \omega_o) L(p' \rightarrow -\omega_i) \cos(\omega_i, n_p) d\omega_i \]

- Area form

\[ L(p \rightarrow \omega_o) = L_e(p \rightarrow \omega_o) + \int_S f_r(p, \omega_i \leftrightarrow \omega_o) L(x \rightarrow -\omega_i) V(p, x) \frac{\cos(\omega_i, n_p) \cos(-\omega_i, n_x)}{r^2_{px}} dA_x \]
Motivation - Monte Carlo Integration

- Choose an appropriate probability density function
- Generate random samples according to the PDF
- Evaluate the function for all samples
- Accumulate function values weighted with their approximate sample size
Inversion Method

- Mapping of a uniform random variable to a goal distribution
- Discrete example
  - Four outcomes with probabilities $p_1, p_2, p_3, p_4$ and $\sum_i p_i = 1$
  - Computation of the cumulative distribution function $P(i) = \sum_{j=1}^i p_j$

[Diagram showing the cumulative distribution function for four outcomes with probabilities $p_1, p_2, p_3, p_4$.]

[Pharr, Humphreys]
Inversion Method

- Discrete example cont.
  - Take a uniform random variable $\xi$
  - $P^{-1}(\xi)$ has the desired distribution

- Continuous case
  - $P$ and $P^{-1}$ are continuous functions
  - Start with the desired PDF $p(x)$
  - Derive $P(x) = \int_0^x p(x')dx'$
  - Compute the inverse $P^{-1}(x)$
  - Obtain a uniformly distributed variable
  - Compute $X_i = P^{-1}(\xi)$ which adheres to $p(x)$
Inversion Method - Example 1

- Power distribution $p(x) \propto x^n$
  - E.g., for sampling the Blinn microfacet model
- Computation of the PDF
  - $\int_0^1 c \cdot x^n \, dx = 1 \Rightarrow c \left[ \frac{x^{n+1}}{n+1} \right]_0^1 = 1 \Rightarrow c = n + 1$
- PDF $p(x) = (n + 1)x^n$
- CDF $P(x) = \int_0^x p(x') \, dx' = x^{n+1}$
- Inverse of the CDF $P^{-1}(x) = x^{n+1}$
- Sample generation
  - Generate uniform random samples $0 \leq \xi \leq 1$
  - $X = \frac{n+1}{\sqrt[3+\sqrt{\xi}]{\xi}}$ are samples from the distribution $p(x) = (n + 1)x^n$
Inversion Method - Example 2

- Exponential distribution \( p(x) \propto e^{-ax} \)
  - E.g., for considering participating media
- Computation of the PDF
  - \( \int_0^\infty c \ e^{-ax} \, dx = -\frac{c}{a} \ e^{-ax} \bigg|_0^\infty = \frac{c}{a} = 1 \)
- PDF \( p(x) = a \ e^{-ax} \)
- CDF \( P(x) = \int_0^x p(x') \, dx' = 1 - e^{-ax} \)
- Inverse of the CDF \( P^{-1}(x) = \frac{-\ln(1-x)}{a} \)
- Sample generation
  - Generate uniform random samples \( 0 \leq \xi \leq 1 \)
  - \( X = -\frac{\ln(1-\xi)}{a} \) are samples from the distribution \( p(x) = a \ e^{-ax} \)
Outline

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Rejection Method

- Draws samples according to a function $f(x)$
  - Dart-throwing approach
  - Works with a PDF $p(x)$ and a scalar $c$ with $f(x) < c \cdot p(x)$
- Properties
  - $f(x)$ is not necessarily a PDF
  - PDF, CDF and inverse CDF do not have to be computed
  - Simple to implement
  - Useful for debugging purposes
Rejection Method

- Sample generation
  - Generate a uniform random sample $0 \leq \xi < 1$
  - Generate a sample $X$ according to $p(x)$
  - Accept $X$ if $\xi \cdot c \cdot p(X) \leq f(X)$

[Pharr, Humphreys]
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Transforming Between Distributions

- Computation of a resulting PDF, when a function is applied to samples from an arbitrary distribution
  - Random variables $X_i$ are drawn from $p_x(x)$
  - Bijective transformation (one-to-one mapping) $Y_i = y(X_i)$
  - How does the distribution $p_y(y)$ look like?
Transforming Between Distributions

- \( Pr\{Y \leq y(x)\} = Pr\{X \leq x\} \)

\[ P_y(y) = P_y(y(x)) = P_x(x) \]

\[ p_y(y) = \frac{p_x(x)}{|y'(x)|} \]

- Example \( p_x(x) = 2x \) \( 0 \leq x \leq 1 \)
  - \( y(x) = \sin x \) \( x(y) = \arcsin y \)
  - \( y'(x) = \cos x \)
  - \( p_y(y) = \frac{p_x(x)}{|\cos x|} = \frac{2x}{|\cos x|} = \frac{2 \arcsin y}{|\cos \arcsin(y)|} = \frac{2 \arcsin y}{\sqrt{1-y^2}} \)
Transforming Between Distributions

- Multiple dimensions
  - $X_i$ is an n-dimensional random variable
  - $Y_i = T(X_i)$ is a bijective transformation
- Transformation of the PDF

$$p_y(y) = \frac{p_x(x)}{|J_T(x)|} \quad J_T(x) = \begin{pmatrix}
\frac{\partial T_1}{\partial x_1} & \cdots & \frac{\partial T_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial T_n}{\partial x_1} & \cdots & \frac{\partial T_n}{\partial x_n}
\end{pmatrix}$$
Transforming Between Distributions

- Example (polar coordinates)
  - Samples \((r, \theta)\) with density \(p(r, \theta)\)
  - Corresponding density \(p(x, y)\) with \(x = r \cos \theta\) and \(y = r \sin \theta\)

\[
J_T(x) = \begin{pmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{pmatrix}
\]

\(|J_T(x)| = r(\cos^2 \theta + \sin^2 \theta) = r\)

- \(p(x, y) = \frac{1}{r}p(r, \theta)\)
  \(p(r, \theta) = r \cdot p(x, y)\)
Transforming Between Distributions

– Example (spherical coordinates)
  – \( x = r \sin \theta \cos \phi \)
  – \( y = r \sin \theta \sin \phi \)
  – \( z = r \cos \theta \)
  – \( p(r, \theta, \phi) = r^2 \sin \theta \cdot p(x, y, z) \)

– Example (solid angle)
  – \( Pr\{\omega \in \Omega\} = \int_{\Omega} p(\omega) d\omega \)
  – \( d\omega = \sin \theta \ d\theta \ d\phi \)
  – \( p(\theta, \phi) = \sin \theta \cdot p(\omega) \)
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Concept

- Samples from a 2D joint density function $p(x, y)$
- Procedure
  - Compute the marginal density function $p_x(x) = \int p(x, y) dy$
  - Compute the conditional density function $p_y(y|x) = \frac{p(x,y)}{p_x(x)}$
  - Generate a sample $X$ according to $p_x(x)$
  - Generate a sample $Y$ according to $p_y(y|X) = \frac{p(x,y)}{p_x(X)}$
- Marginal density function
  - Integral of $p(x, y)$ for a particular $x$ over all $y$-values
- Conditional density function
  - Density function for $y$ given a particular $x$
Outline

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Uniform Sampling of a Triangle

- Sampling an isosceles right triangle of area 0.5
  - $u, v$ can be interpreted as Barycentric coordinates
  - Can be used to generate samples for arbitrary triangles
- $p(u, v) = 2$
- Marginal density function
  - $p_u(u) = \int_0^{1-u} p(u, v) \, dv = 2 \int_0^{1-u} dv = 2(1 - u)$
- Conditional density
  - $p_v(v|u) = \frac{p(u, v)}{p_u(u)} = \frac{1}{1-u}$
- Inversion method
  - $P_u(u) = \int_0^u 2 - 2u' \, du' = 2u - u^2$
  - $P_v(v|u) = \int_0^v \frac{1}{1-u'} \, dv' = \frac{v}{1-u}$

[Pharr, Humphreys]
Uniform Sampling of a Triangle

- Inversion method cont.
  - Inverse functions of the cumulative distribution functions
    - $u = 1 - \sqrt{1 - \xi_1}$ \hspace{1cm} u is generated between 0 and 1
    - $v = \xi_2 \sqrt{1 - \xi_1}$ \hspace{1cm} v is generated between 0 and 1-$u=(1-\xi)^{\frac{1}{2}}$
  - Generating uniformly sampled random values $\xi_1$ and $\xi_2$
  - Applying the inverse CDFs to obtain $u$ and $v$
Uniform Sampling of a Hemisphere

- PDF is constant with respect to a solid angle $p(\omega) = c$

\[- \int_{2\pi} p(\omega) d\omega = 1 \Rightarrow c \int_{2\pi} d\omega = 1 \Rightarrow c = \frac{1}{2\pi}\]

- $p(\omega) = \frac{1}{2\pi} \Rightarrow p(\theta, \phi) = \frac{\sin \theta}{2\pi}$

- Marginal density function
  \[- p_\theta(\theta) = \int_0^{2\pi} p(\theta, \phi) d\phi = \int_0^{2\pi} \frac{\sin \theta}{2\pi} d\phi = \sin \theta\]

- Conditional density for $\phi$
  \[- p_\phi(\phi | \theta) = \frac{p(\theta, \phi)}{p_\theta(\theta)} = \frac{1}{2\pi}\]

- Inversion method
  \[- P_\theta(\theta) = \int_0^\theta \sin \theta' d\theta' = -\cos \theta + 1\]
  \[- P_\phi(\phi | \theta) = \int_0^\phi \frac{1}{2\pi} d\phi' = \frac{\phi}{2\pi}\]
Uniform Sampling of a Hemisphere

– Inversion method cont.
  – Inverse functions of the cumulative distribution functions
    – \( \theta = \arccos(1 - \xi_1) \)
    – \( \phi = 2\pi \xi_2 \)
  – Generating uniformly sampled random values \( \xi_1 \) and \( \xi_2 \)
  – Applying the inverse CDFs to obtain \( \theta \) and \( \phi \)

– Conversion to Cartesian space
  – \( x = \sin \theta \cos \phi = \cos(2\pi \xi_2) \sqrt{1 - (1 - \xi_1)^2} \)
  – \( y = \sin \theta \sin \phi = \sin(2\pi \xi_2) \sqrt{1 - (1 - \xi_1)^2} \)
  – \( z = \cos \theta = 1 - \xi_1 \)

– \((x, y, z)^T\) is a normalized direction
Uniform Sampling of a Hemisphere

Illustration for $\theta$

$\theta = \arccos(1 - \xi_1)$

Generate less samples for smaller angles $\theta$
Cosine-Weighted Sampling of a Hemisphere

- PDF is proportional to \( \cos \theta \): 
  \[ p(\omega) \propto \cos \theta \]
  
  \[
  \int_{2\pi} p(\omega) \, d\omega = 1 = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} c \cos \theta \sin \theta \, d\theta \, d\phi = c \, 2\pi \int_0^{\frac{\pi}{2}} \cos \theta \sin \theta \, d\theta = c \, 2\pi \frac{1}{2} = 1
  \]

- Marginal density function

  \[
  p_\theta(\theta) = \int_0^{2\pi} p(\theta, \phi) \, d\phi = \int_0^{2\pi} \frac{1}{\pi} \cos \theta \sin \theta \, d\phi = 2 \cos \theta \sin \theta
  \]

- Conditional density for \( \phi \)

  \[
  p_\phi(\phi|\theta) = \frac{p(\theta, \phi)}{p_\theta(\theta)} = \frac{1}{2\pi}
  \]

- Inversion method

  \[
  P_\theta(\theta) = \int_0^\theta 2 \cos \theta' \sin \theta' \, d\theta' = 2 \left[ -\frac{\cos^2 \theta'}{2} \right]_0^\theta
  \]

  \[
  = 2 \left( -\frac{\cos^2 \theta}{2} + \frac{1}{2} \right) = \sin^2 \theta
  \]

  \[
  P_\phi(\phi|\theta) = \int_0^\phi \frac{1}{2\pi} \, d\phi' = \frac{\phi}{2\pi}
  \]

[Suffern]
Cosine-Weighted Sampling of a Hemisphere

- Inversion method cont.
  - Inverse functions of the cumulative distribution functions
  - $\theta = \arcsin(\sqrt{\xi_1})$
  - $\phi = 2\pi \xi_2$
  - Generating uniformly sampled random values $\xi_1$ and $\xi_2$
  - Applying the inverse CDFs to obtain $\theta$ and $\phi$
- Conversion to Cartesian space
  - $x = \sin \theta \cos \phi = \cos(2\pi \xi_2) \sqrt{\xi_1}$
  - $y = \sin \theta \sin \phi = \sin(2\pi \xi_2) \sqrt{\xi_1}$
  - $z = \cos \theta = \sqrt{1 - \xi_1}$
  - $(x, y, z)^T$ is a normalized direction

x- y- values uniformly sample a unit disk, i.e., cosine-weighted samples of the hemisphere can also be obtained by uniformly sampling a unit sphere and projecting the samples onto the hemisphere
Cosine-Weighted Sampling of a Hemisphere

Illustration for $\theta$

Generate less samples for smaller and larger angles $\theta$

$\theta = \arcsin(\xi_1)$

Cosine-weighted hemisphere (top view, side view)