Advanced Computer Graphics
Stochastic Raytracing

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Outline

– Context
– Diffuse vs. general global illumination
– Monte Carlo integration
– Sampling of random variables
Context

- Radiosity equation governs light transport for diffuse surfaces. ⇒ How to describe light transport for general surfaces?
- How to solve for the light transport?
- How to compute the relevant part of the light transport towards a sensor?
Context

- Light transport towards the sensor requires to solve
  \[ L(p \rightarrow \omega_o) = \int_S f_r(p, \omega_i \leftrightarrow \omega_o) L(p' \rightarrow -\omega_i) G(p, p') dA_{p'} \]

- Monte Carlo integration approximates this integral
  - E.g., \( L(p \rightarrow \omega_o) \approx \sum_i f_r(p, \omega_i \leftrightarrow \omega_o) L(p' \rightarrow -\omega_i) G(p, p') A_{p'} \)
  - Send rays into the hemisphere
  - Associate an area / solid angle with each ray
  - Compute radiance along this ray
  - Sum up all contributions
Outline

– Context
– Diffuse vs. general global illumination
– Monte Carlo integration
– Sampling of random variables
Governing Equations

- Rendering equation
  - Governing equation for general global illumination methods
    - \( L(p \rightarrow \omega_o) = L_e(p \rightarrow \omega_o) + \)
    \[
    \int_S f_r(p, \omega_i \leftrightarrow \omega_o) L(x \rightarrow -\omega_i)V(p, x) \frac{\cos(\omega_i,n_p)\cos(-\omega_i,n_x)}{r_{px}^2} \, dA_x
    \]
- Radiosity equation
  - Governing equation for diffuse global illumination methods
    - \( L(p \rightarrow \omega_o) = \frac{B(p)}{\pi} \)
    \[
    f_r(p, \omega_i \leftrightarrow \omega_o) = \frac{\rho(p)}{\pi}
    \]
    - \( B(p) = B_e(p) + \frac{\rho(p)}{\pi} \int_S B(x)V(p, x) \frac{\cos(\omega_i,n_p)\cos(-\omega_i,n_x)}{r_{px}^2} \, dA_x \)
A Solution Strategy (Radiosity)

- Finite Element Method (FEM)
- Start with a continuous form / function

\[
B(p) = B_e(p) + \frac{\rho(p)}{\pi} \int_S B(x)V(p, x) \frac{\cos(\omega_i, n_p) \cos(-\omega_i, n_x)}{r_{px}^2} \, dA_x
\]

- Discretization

\[
B = B_e + FB
\]
\[
B = (I - F)^{-1} B_e
\]

- Solving for a vector with unknown radiosities

\[
(I - F)^{-1} = \sum_{k=0}^{\infty} F^k
\]
\[
B = B_e + FB_e + FFB_e + FFFB_e + \ldots
\]
An Alternative Strategy

- Start with the general form of the rendering equation, e.g. in hemispherical form

\[ L(p \rightarrow \omega_o) = L_e(p \rightarrow \omega_o) + \int_{\Omega} f_r(p, \omega_i \leftrightarrow \omega_o) L(p \leftarrow \omega_i) \cos(\omega_i, n_p) d\omega_i \]

- Solving for a function with unknown radiances \( L(p \rightarrow \omega_o) \)
  - I.e., radiance at all surface positions into all directions
Operator Form of the Rendering Equation

- A linear operator transforms a function into another one
- Scattering operator
  - \((Kh)(p \rightarrow \omega_o) = \int_{\Omega} f_r(p, \omega_i \leftrightarrow \omega_o) h(p \leftarrow \omega_i) \cos(\omega_i, n_p) d\omega_i\)
  - Applied to an incident radiance function \(L(p \leftarrow \omega_i)\), exitant radiance resulting from one scattering step is returned
  - \(L(p \rightarrow \omega_o) = (KL)(p \leftarrow \omega_i)\)
  - \(K\) operates on an entire function, i.e. on all incident radiances for all positions \(p\) and direction \(\omega_i\)

Operator Form of the Rendering Equation

- Propagation operator
  - \((Gh)(p \leftarrow \omega_i) = h(p' \rightarrow -\omega_i)\) \(p'\) indicates the raycast operator applied to \(p\)
  - Applied to an exitant radiance function \(L(p' \rightarrow -\omega_i)\), incident radiance at \(p\) from direction \(\omega_i\) is returned
  - \(L(p \leftarrow \omega_i) = (GL)(p' \rightarrow -\omega_i)\)
  - Radiance is preserved / propagated along the line between \(p\) and \(p'\)
  - \(p\) and \(p'\) can be reversed, i.e. \(L(p' \leftarrow -\omega_o) = (GL)(p \rightarrow \omega_o)\)

Operator Form of the Rendering Equation

- Light transport operator
  - $T = KG$
  - Composition of scattering and propagation
  - Maps an exitant radiance function the exitant radiance function after one scattering step
  - Remember: $G$ maps exitant radiance to incident radiance propagated along a direction. Then, $K$ maps incident radiance to exitant radiance after scattering

Operator Form of the Rendering Equation

- \( L(p \rightarrow \omega_o) = L_e(p \rightarrow \omega_o) + \int_{\Omega} f_r(p, \omega_i \leftrightarrow \omega_o) L(p \leftarrow \omega_i) \cos(\omega_i, \mathbf{n}_p) d\omega_i \)

- Can be written as

  \[ L(p \rightarrow \omega_o) = L_e(p \rightarrow \omega_o) + (KGL)(p \rightarrow \omega_o) \]

- Light transport equation
  - \( L = L_e + TL \) Infinite number of equations with an infinite number of unknown exitant radiances
  - Relates exitant radiance functions
  - Represents the light propagation equilibrium
Light Transport Equation

- $L = L_e + TL$
- Solving for the unknown radiance function
  - $(I - T)L = L_e$
  - $L = (I - T)^{-1}L_e$
  - Neumann series
  - $L = \sum_{k=0}^{\infty} (T^k L_e)$
    $\approx L_e + TL_e + T^2 L_e + T^3 L_e + \ldots$
Light Transport Equation

- **Discussion**
  - Radiance function is linear with respect to emission
    \[ L = (I - T)^{-1}(L_{e,1} + L_{e,2}) = (I - T)^{-1}L_{e,1} + (I - T)^{-1}L_{e,2} \]
  - Radiance function is a sum of
    - Emitted radiance \( L_e \)
    - Emitted radiance after one reflection \( TL_e \)
    - Emitted radiance after two reflections \( TTL_e \)
    - ...
    - \( L \approx L_e + TL_e + TTL_e + TTTL_e + \ldots \)
Terms in the Neumann Series

- Example contributions to terms
Forward Raytracing

- Send rays / propagate radiance from all light source positions into all directions $\Rightarrow L_e$

- At all intersection points $p$, solve the integral
  \[ L_1(p \rightarrow \omega_o) = \int_\Omega f_r(p, \omega_i \leftrightarrow \omega_o)GL_e \cos(\omega_i, n_p) d\omega_i \]
  for all direction $\omega_o \Rightarrow TL_e$

- Send rays to propagate $TL_e$

- At all intersection points $p$, solve the integral
  \[ L_2(p \rightarrow \omega_o) = \int_\Omega f_r(p, \omega_i \leftrightarrow \omega_o)GL_1 \cos(\omega_i, n_p) d\omega_i \]
  for all direction $\omega_o \Rightarrow TTL_e$
Forward Raytracing

At a sensor: Accumulate radiance contributions of rays after n scattering steps, i.e. compute \( L_e + TL_e + T^2L_e + \ldots \)
Backward Raytracing

- Send rays from the sensor into the scene
- Propagate radiance from visible light sources
- $\Rightarrow$ part of $L_e$ visible to the sensor
- At intersection points $p$ with the scene, compute radiance $L(p \rightarrow \omega_o) = \int_{\Omega} f_r(p, \omega_i \leftrightarrow \omega_o) L_e(p \leftarrow \omega_i) \cos(\omega_i, n_p) d\omega_i$ that is propagated in direction $\omega_o$ towards the sensor
- $\Rightarrow$ part of $T L_e$ visible to the sensor
- ...
Backward Raytracing

– Trace rays from the sensor into the scene
Setting at Sensor

- How to compute $L(p_1 \leftarrow \omega_o)$ and what is its the relation to $L_e + TL_e + TTTL_e + \ldots$
Setting at First-Level Intersections

\[ L(p_1 \rightarrow \omega_o) = \int_S f_r(p_1, \omega_i \leftrightarrow \omega_o) L(p_2 \rightarrow -\omega_i) G(p_1, p_2) dA_{p_2} \]

\[ = \int_{\text{Light Sources}} f_r(p_1, \omega_i \leftrightarrow \omega_o) L_e(p_2 \rightarrow -\omega_i) G(p_1, p_2) dA_{p_2} \]

\[ + \int_{\text{Scene}} f_r(p_1, \omega_i \leftrightarrow \omega_o) L(p_2 \rightarrow -\omega_i) G(p_1, p_2) dA_{p_2} \]

- \( \int_{\text{Light Sources}} \cdots \) is the part of \( TL_e \) visible to the sensor

- Computation of \( \int_{\text{Scene}} \cdots \) requires \( L(p_2 \rightarrow -\omega_i) \)
Setting at Second-Level Intersections

- \( L(p_2 \rightarrow \omega_o) = \int_S f_r(p_2, \omega_i \leftrightarrow \omega_o) L(p_3 \rightarrow -\omega_i) G(p_2, p_3) dA_{p_3} \)
  = \( \int_{\text{Light Sources}} f_r(p_2, \omega_i \leftrightarrow \omega_o) L_e(p_3 \rightarrow -\omega_i) G(p_2, p_3) dA_{p_3} \)
  + \( \int_{\text{Scene}} f_r(p_2, \omega_i \leftrightarrow \omega_o) L(p_3 \rightarrow -\omega_i) G(p_2, p_3) dA_{p_3} \)

- \( \int_{\text{Light Sources}} \cdots \) is the part of \( TTTL_e \) visible to the sensor

- Computation of \( \int_{\text{Scene}} \cdots \) requires \( L(p_3 \rightarrow -\omega_i) \)

\[ L(p_2 \leftarrow \omega_i) = L(p_3 \rightarrow -\omega_i) \]

Towards \( p_1 \)
Summary

– Recursive evaluation of

\[ L(p \to \omega_o) = \int_S f_r(p, \omega_i \leftrightarrow \omega_o) L(p' \to -\omega_i) G(p, p') dA_{p'} \]

\[ = \int_{\text{Light Sources}} f_r(p, \omega_i \leftrightarrow \omega_o) L_e(p' \to -\omega_i) G(p, p') dA_{p'} + \int_{\text{Scene}} f_r(p, \omega_i \leftrightarrow \omega_o) L(p' \to -\omega_i) G(p, p') dA_{p'} \]

– Each recursion level computes parts of the functions \( L_e, TL_e, TT L_e, \ldots \) that are visible to the sensor
Numerical Integration

- The integral $\int_S \ldots$ is approximately computed with a sum of samples $\sum_i \ldots$
- For each sample $i$,
  - A ray is cast into the scene
  - Intersection with the scene is computed
  - Radiance along the ray is computed
Numerical Integration

- Typically, $\int_S \ldots = \int_{\text{Scene}} \ldots + \int_{\text{Light Sources}} \ldots \approx \sum_{\text{Scene}_i} \ldots + \sum_{\text{Light Source}_i} \ldots$ is considered
- For $\sum_{\text{Light Source}_i} \ldots$, light source areas are sampled and rays towards those positions are processed
- For $\sum_{\text{Scene}_i} \ldots$, the respective solid angle is sampled and rays towards those directions are processed
Numerical Integration

− Due to the recursive nature, the number of processed rays grows exponentially with the recursion level
− ⇒ Monte Carlo integration
  − Efficient for multidimensional integral
  − Very flexible in terms of the number of used samples
  − Adaptive sample distribution
  − Even one sample can be used to approximate an integral
  − ⇒ e.g., Path tracing
    − At each recursion level, send a fixed number of rays to light sources and one ray into the scene (which generates a ray path)
Outline

– Context
– Diffuse vs. general global illumination
– Monte Carlo integration
– Sampling of random variables
Goal

– Approximating the solution of the light transport equation \( L = \sum_{k=0}^{\infty} (T^k L_e) \)

– Recursive evaluation of

\[
L(p \to \omega_o) = \int_S f_r(p, \omega_i \leftrightarrow \omega_o) L(p' \to -\omega_i) G(p, p') dA_{p'}
\]

\[
= \int_{\text{Light Sources}} f_r(p, \omega_i \leftrightarrow \omega_o) L_e(p' \to -\omega_i) G(p, p') dA_{p'}
\]

\[
+ \int_{\text{Scene}} f_r(p, \omega_i \leftrightarrow \omega_o) L(p' \to -\omega_i) G(p, p') dA_{p'}
\]

– Each recursion level computes parts of the functions \( L_e, TL_e, T^2 L_e, \ldots \) that are visible to the sensor
Numerical Integration – Fixed Sample Size

- E.g. Riemann sum
  \[ \int_a^b f(x)\,dx \approx \sum_i f(x_i)\Delta x \quad \Delta x = \frac{b-a}{N} \]
- More / smaller samples \( \Rightarrow \) better accuracy
- \( d \) dimensional integrals require \( N^d \) samples
Numerical Integration – Adaptive Sample Size

– E.g., Monte Carlo integration
  \[ \int_a^b f(x) \, dx \approx \sum_i f(x_i) \Delta x_i, \text{ adaptive sample size } \Delta x_i \]
– More / smaller samples ⇒ better accuracy
– \( d \) dimensional integrals work with arbitrary sample numbers
– Sample size is only approximated ⇒ noise
Stochastic Raytracing - Concept

- Approximately evaluate the integral
\[ \int_{\Omega} f_r(p, \omega_i \leftrightarrow \omega_o) L(p \leftarrow \omega_i) \cos(\omega_i, n_p) d\omega_i \]
  by
  - Tracing rays into randomly sampled 2D directions
  - Computing the incoming radiances

- Integral is approximated with
\[ \sum_i f_r(p, \omega_i \leftrightarrow \omega_o) L(p \leftarrow \omega_i) \cos(\omega_i, n_p) \Delta \Omega_i \]
  - 2 dimensional sample directions \( \omega_i = (\theta_i, \phi_i) \)
  - \( \Delta \Omega_i \) is an approximation of the solid angle of sample direction \( \omega_i = (\theta_i, \phi_i) \)
Introduction

– Challenges
  – Approximate the integral as exact as possible
  – Trace as few rays as possible / use as few samples as possible
  – Trace relevant rays / use relevant samples
    – Rays / samples to light sources are very relevant
    – For diffuse surfaces, rays / samples in normal direction are more relevant than rays / samples perpendicular to the normal
    – For specular surfaces, rays / samples in reflection direction are relevant
Properties

- Benefits
  - Processes only evaluations of the integrand at arbitrary surface points in the domain
  - Appropriate for integrals of arbitrary dimensions
  - Allows for non-uniform sample patterns / adaptive sample sizes
  - Works for a large variety of integrands, e.g., it handles discontinuities
Properties

- Drawbacks
  - Using n samples, the scheme converges to the correct result with $O(n^{1/2})$
  - I.e., to half the error, 4n samples are required
  - Errors are perceived as noise, i.e. pixels are arbitrarily too bright or dark (due to the erroneous approximation of the sample size)
  - Evaluation of the integrand at a point is expensive (ray intersections tests)
Continuous Random Variables

– Motivation: Random sampling of directions
– Continuous random variables $x$
  – In contrast to discrete random variables, infinite number of possible values
– Canonical uniform random variable $0 \leq \xi < 1$
  – Sample sets with arbitrary distributions can be computed from $\xi$
**Probability Density Function PDF \( p(x) \)**

– Motivation: PDF governs the size / solid angle of a sample / sample direction
– Probability of a random variable taking certain value ranges
– \( p(x) \geq 0 \quad \forall x \in [a, b] \)
– \( \int_a^b p(x) \, dx = 1 \)
– \( Pr(x_0 \leq X \leq x_1) = \int_{x_0}^{x_1} p(x) \, dx \)
– Example
  – Uniform PDF for \( 0 \leq X \leq 5 \)
  – \( 1 = \int_0^5 p(x) \, dx = p(x) \int_0^5 \, dx = 5 \, p(x) \)
  – \( p(x) = \frac{1}{5} \)

The probability, that the random variable is in the specified domain, is 1.

The probability, that the random variable has a certain exact value \( x_0=x_1 \), is 0.
Cumulative Distribution Function CDF \( P(x) \)

- Motivation: CDFs are required to generate sample sets for arbitrary PDFs from uniform sample sets
- Probability of a random variable to be less or equal to \( x \)
  
  \[
  P(x) = Pr(X \leq x) = \int_a^x p(x)dx
  \]
  
  \[
  P(a) = 0 \leq P(x) \leq 1 = P(b)
  \]
  
  \[
  Pr(x_0 \leq X \leq x_1) = P(x_1) - P(x_0)
  \]
Expected Value

– Motivation: expected value of an estimator function is equal to the integral in the reflectance equation

– Expected value \( E_p[f(x)] \) of a function \( f(x) \) is defined as the weighted average value of the function over a domain \( D \)

\[
E_p[f(x)] = \int_D f(x) \ p(x) \ dx \quad \text{with} \quad \int_D p(x) \ dx = 1
\]

– Properties

\[
\begin{align*}
E[af(x)] &= aE[f(x)] \\
E[\sum_i f(X_i)] &= \sum_i E[f(X_i)]
\end{align*}
\]

For independent random variables \( X_i \)

Processes an infinite number of samples \( x \) according to a PDF \( p(x) \)
**Expected Value**

- Examples for uniform PDF \( p(x) \)
  - \( f(x) = \cos(x) \) \( D = [0, \pi] \) \( p(x) = \frac{1}{\pi} \)
    \[
    E_p[\cos(x)] = \int_0^{\pi} \cos(x) \frac{1}{\pi} \, dx = \frac{1}{\pi} (-\sin \pi + \sin 0) = 0
    \]
  - \( f(x) = x \) \( D = [0, 6] \) \( p(x) = \frac{1}{6} \)
    \[
    E_p[x] = \int_0^6 x \frac{1}{6} \, dx = \frac{1}{6} (\frac{6^2}{2} - 0) = 3
    \]
  - \( f(x) \)
    \[
    E_p[f(x)] = \frac{1}{b-a} \int_a^b f(x) \, dx
    \]
    \[
    \int_a^b f(x) \, dx = E_p[f(x)](b - a)
    \]
Monte Carlo Estimator - Uniform Random Variables

- Motivation: approximation of the integral in the reflectance equation
- Goal: computation of \( \int_{a}^{b} f(x) \, dx \)
- Uniformly distributed random variables \( X_i \in [a, b] \)
- Probability density function \( p(x) = \frac{1}{b-a} \) Constant and integration to one
- Monte Carlo estimator \( F_N = \frac{b-a}{N} \sum_{i=1}^{N} f(X_i) \)
- Expected value of \( F_N \) is equal to the integral \( \int_{a}^{b} f(x) \, dx \)
  \[ E[F_N] = \int_{a}^{b} f(x) \, dx \]
Monte Carlo Estimator - Uniform Random Variables

\[ E[F_N] = E \left[ \frac{b-a}{N} \sum_{i=1}^{N} f(X_i) \right] \]

\[ = \frac{b-a}{N} \sum_{i=1}^{N} E[f(X_i)] \]

\[ = \frac{b-a}{N} \sum_{i=1}^{N} \int_{a}^{b} f(x)p(x)dx \]

\[ = \frac{b-a}{N} \sum_{i=1}^{N} \int_{a}^{b} f(x) \frac{1}{b-a} dx \]

\[ = \frac{1}{N} \sum_{i=1}^{N} \int_{a}^{b} f(x)dx \]

\[ = \int_{a}^{b} f(x)dx \]
Monte Carlo Estimator - Uniform Random Variables

- PDF \( p(x) = \frac{1}{b-a} \)
- Estimator \( F_N = \frac{b-a}{N} \sum_{i=1}^{N} f(X_i) \)
- Integral
  - \( \int_{a}^{b} f(x)dx \approx \frac{b-a}{N} \sum_{i=1}^{N} f(X_i) = \sum_{i=1}^{N} f(X_i) \frac{b-a}{N} = \sum_{i=1}^{N} f(X_i) \frac{1}{N \ p(X_i)} \)
- Function value \( f(X_i) \)
- Approximate sample size \( \frac{1}{N \ p(X_i)} \)
Examples - Uniform Random Variables

- Integral \( \int_0^1 5x^4 \, dx = 1 \)
- Estimator \( F_N = \frac{1-0}{N} \sum_{i=1}^{N} 5X_i^4 \)  
  Sample size approx. \( 1/N \)
- For an increasing number of uniformly distributed random variables \( X_i \), the estimator converges to one:
  \[
  F_N = \frac{b-a}{N} \sum_{i=1}^{N} f(X_i) \\
  F_N = (b-a) \frac{1}{N} \sum_{i=1}^{N} f(X_i) \\
  F_N = (b-a) f(x) \\
  E[F_N] = \int_{a}^{b} f(x) \, dx
  \]
Monte Carlo Estimator - Non-uniform Random Variables

- Monte Carlo estimator

\[ F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)} \quad \text{for} \quad p(X_i) \neq 0 \]

- \[ E[F_N] = E \left[ \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)} \right] \]

\[ = \frac{1}{N} \sum_{i=1}^{N} \int_a^b \frac{f(x)}{p(x)} p(x) dx \]

\[ = \frac{1}{N} \sum_{i=1}^{N} \int_a^b f(x) dx \]

\[ = \int_a^b f(x) dx \]
Monte Carlo Estimator - Non-uniform Random Variables

- PDF $p(x)$
- Estimator $F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)}$
- Integral
  - $\int_{a}^{b} f(x) \, dx \approx \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)} = \sum_{i=1}^{N} f(X_i) \frac{1}{N \cdot p(X_i)}$
  - Function value $f(X_i)$
  - Approximate sample size $\frac{1}{N \cdot p(X_i)}$
Approximate Sample Size

- Sample size / distance for uniform PDF: \( \approx \frac{b-a}{N} = \frac{1}{Np(X_i)} \)

- Sample size for non-uniform PDF: \( \approx \frac{1}{Np(X_i)} \)
Monte Carlo Estimator - Multiple Dimensions

- Samples $X_i$ are multidimensional
- E.g., $\int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} f(x, y, z) dx dy dz$
- Uniformly distributed random samples
  $\{x_0, y_0, z_0\} \leq X_i = (x_i, y_i, z_i) \leq \{x_1, y_1, z_1\}$
- Probability density function $p(X_i) = \frac{1}{x_1 - x_0} \frac{1}{y_1 - y_0} \frac{1}{z_1 - z_0}$
- Monte Carlo estimator
  $F_N = \frac{(x_1 - x_0)(y_1 - y_0)(z_1 - z_0)}{N} \sum_{i=1}^{N} f(X_i)$
- Approximate sample volume is $\frac{(x_1 - x_0)(y_1 - y_0)(z_1 - z_0)}{N}$
Monte Carlo Estimator - Multiple Dimensions

- E.g., \( \int_{1}^{4} \int_{1}^{4} f(x, y) \, dx \, dy \)
- Uniformly distributed random samples
- Probability density function \( p(X_i) = \frac{1}{4-1} \frac{1}{4-1} = \frac{1}{9} \)
- Monte Carlo estimator \( F_N = \frac{9}{N} \sum_{i=1}^{N} f(X_i) \)
- Approximate sample volume \( \frac{9}{N} \)
Monte Carlo Estimator - Integration over a Hemisphere

- Approximate computation of the irradiance at a point
  
  \[ E_i(p) = \int_{2\pi} L_i(p, \omega) \cos \theta d\omega \]
  
  \[ = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} L_i(p, \theta, \phi) \cos \theta \sin \theta \, d\theta \, d\phi \]

- Estimator
  
  \[ F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)} = \frac{1}{N} \sum_{i=1}^{N} \frac{L_i(p, \theta_i, \phi_i) \cos \theta_i \sin \theta_i}{p(\theta_i, \phi_i)} \]

- Choosing a PDF
  
  This flexibility is an important aspect of Monte Carlo integration.
  
  - Should be similar to the shape of the integrand
  - As incident radiance is weighted with \( \cos \theta \), it is appropriate to generate more samples close to the top of the hemisphere
  - \( p(\theta, \phi) \propto \cos \theta \)
Monte Carlo Estimator - Integration over a Hemisphere

- Probability distribution

\[
\int_{2\pi} c \ p(\omega) d\omega = 1 \\
\int_{0}^{2\pi} \int_{0}^{\frac{\pi}{2}} c \ \cos \theta \sin \theta \ d\theta \ d\phi = 1 \\
c \ \frac{2\pi}{1+1} = 1 \\
c = \frac{1}{\pi} \\
p(\theta, \phi) = \frac{\cos \theta \sin \theta}{\pi}
\]

- Estimator

\[
F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{L_i(p, \theta_i, \phi_i) \cos \theta_i \sin \theta_i}{p(\theta_i, \phi_i)} \\
= \frac{\pi}{N} \sum_{i=1}^{N} L_i(p, \theta_i, \phi_i) \approx \int_{0}^{2\pi} \int_{0}^{\frac{\pi}{2}} L_i(p, \theta, \phi) \cos \theta \sin \theta d\theta d\phi
\]

If \(\theta\) and \(\phi\) are sampled according to PDF \(p(\theta, \phi)\)
Monte Carlo Estimator - Integration over a Hemisphere

- Integral \( \int_0^{2\pi} \int_0^{\frac{\pi}{2}} L_i(p, \theta, \phi) \cos \theta \sin \theta \, d\theta \, d\phi \)
- PDF \( p(\theta, \phi) = \frac{\cos \theta \sin \theta}{\pi} \)
- Estimator \( \frac{\pi}{N} \sum_{i=1}^{N} L_i(p, \theta_i, \phi_i) \)
  \[ = \sum_{i=1}^{N} L_i(p, \theta_i, \phi_i) \cos \theta_i \sin \theta_i \frac{\pi}{N \cos \theta_i \sin \theta_i} \]
- Function value \( L_i(p, \theta_i, \phi_i) \cos \theta_i \sin \theta_i \) for direction \((\theta_i, \phi_i)\)
- Approximate sample size / solid angle \( \frac{\pi}{N \cos \theta_i \sin \theta_i} \)
Monte Carlo Integration - Steps

– Choose an appropriate probability density function
– Generate random samples according to the PDF
– Evaluate the function for all samples
– Accumulate sample values weighted with their approximate sample size
- Importance sampling
  - Motivation: contributions of larger sample values are more important
  - PDF should be similar to the shape of the function
  - Optimal PDF \( p(x) = \frac{f(x)}{\int f(x)dx} \)
  - E.g., if incident radiance is weighted with \( \cos \theta \), the PDF should choose more samples close to the normal direction
Monte Carlo Estimator - Variance / Error Reduction

– Stratified sampling
  – Domain subdivision into strata
  – E.g., handling direct and indirect illumination differently

\[ L(p \rightarrow \omega_o) = \int_S f_r(p, \omega_i \leftrightarrow \omega_o) L(p' \rightarrow -\omega_i) G(p, p') dA_{p'} \]

\[ = \int_{\text{Light Sources}} f_r(p, \omega_i \leftrightarrow \omega_o) L_e(p' \rightarrow -\omega_i) G(p, p') dA_{p'} + \int_{\text{Scene}} f_r(p, \omega_i \leftrightarrow \omega_o) L(p' \rightarrow -\omega_i) G(p, p') dA_{p'} \]
Goal

- Approximating the solution of the light transport equation $L = \sum_{k=0}^{\infty} (T^k L_e)$
- Recursive evaluation of

$$L(p \rightarrow \omega_o) = \int_S f_r(p, \omega_i \leftrightarrow \omega_o) L(p' \rightarrow -\omega_i) G(p, p') \, dA_{p'}$$

$$= \int_{\text{Light Sources}} f_r(p, \omega_i \leftrightarrow \omega_o) L_e(p' \rightarrow -\omega_i) G(p, p') \, dA_{p'}$$

$$+ \int_{\text{Scene}} f_r(p, \omega_i \leftrightarrow \omega_o) L(p' \rightarrow -\omega_i) G(p, p') \, dA_{p'}$$

- Each recursion level computes parts of the functions $L_e, TL_e, T^2L_e, \ldots$ that are visible to the sensor
Monte Carlo Estimator - Non-uniform Random Variables

- PDF $p(x)$
- Estimator $F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)}$
- Integral
  - $\int_{a}^{b} f(x) \, dx \approx \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)} = \sum_{i=1}^{N} f(X_i) \frac{1}{N \cdot p(X_i)}$
- Function value $f(X_i)$
- Approximate sample size $\frac{1}{N \cdot p(X_i)}$
Outline

– Context
– Diffuse vs. general global illumination
– Monte Carlo integration
– Sampling of random variables
  – Inversion method
  – Rejection method
  – Transforming between distributions
  – 2D sampling
  – Examples
Inversion Method

- Mapping of a uniform random variable to a goal distribution
- Discrete example
  - Four outcomes with probabilities $p_1, p_2, p_3, p_4$ and $\sum_i p_i = 1$
  - Computation of the cumulative distribution function $P(i) = \sum_{j=1}^{i} p_j$

[Pharr, Humphreys]
Inversion Method

- Discrete example cont.
  - Take a uniform random variable $\xi$
  - $P^{-1}(\xi)$ has the desired distribution

- Continuous case
  - $P$ and $P^{-1}$ are continuous functions
  - Start with the desired PDF $p(x)$
  - Compute $P(x) = \int_0^x p(x')dx'$
  - Compute the inverse $P^{-1}(x)$
  - Obtain a uniformly distributed variable
  - Compute $X_i = P^{-1}(\xi)$ which adheres to $p(x)$

[Pharr, Humphreys]
Inversion Method - Example 1

- Power distribution $p(x) \propto x^n$
  - E.g., for sampling the Blinn microfacet model
- Computation of the PDF
  - $\int_0^1 c \ x^n \, dx = 1 \Rightarrow c \frac{x^{n+1}}{n+1} \bigg|_0^1 = 1 \Rightarrow c = n + 1$
- PDF $p(x) = (n + 1)x^n$
- CDF $P(x) = \int_0^x p(x') \, dx' = x^{n+1}$
- Inverse of the CDF $P^{-1}(x) = \sqrt[n+1]{x}$
- Sample generation
  - Generate uniform random samples $0 \leq \xi \leq 1$
  - $X = \sqrt[n+1]{\xi}$ are samples from the distribution $p(x) = (n + 1)x^n$
Inversion Method - Example 2

- Exponential distribution \( p(x) \propto e^{-ax} \)
  - E.g., for considering participating media
- Computation of the PDF
  - \( \int_0^\infty c \ e^{-ax} \, dx = -\frac{c}{a} \ e^{-ax} \bigg|_0^\infty = \frac{c}{a} = 1 \)
- PDF \( p(x) = a \ e^{-ax} \)
- CDF \( P(x) = \int_0^x p(x') \, dx' = 1 - e^{-ax} \)
- Inverse of the CDF \( P^{-1}(x) = -\frac{\ln(1-x)}{a} \)
- Sample generation
  - Generate uniform random samples \( 0 \leq \xi \leq 1 \)
  - \( X = -\frac{\ln(1-\xi)}{a} \) are samples from the distribution \( p(x) = a \ e^{-ax} \)
Inversion Method - Example 3

- Piecewise-constant distribution
  - E.g., for environment lighting
    \[
    f(x) = \begin{cases} 
    v_0 & x_0 \leq x < x_1 \\
    v_1 & x_1 \leq x < x_2 \\
    \vdots & \vdots \\
    \end{cases}
    \]
    \[x_i = \Delta \cdot i\]
    \[\Delta = \frac{1}{N}\]

- PDF
  \[p(x) = \frac{1}{c} f(x)\]
  with
  \[c = \int_0^1 f(x) \, dx = \sum_{i=0}^{N-1} \Delta \cdot v_i = \sum_{i=0}^{N-1} \frac{v_i}{N}\]

[Pharr, Humphreys]
Inversion Method - Example 3

- CDF \( P(x_0) = 0 \)
  \[
P(x_1) = \int_{x_0}^{x_1} p(x) \, dx = \Delta \cdot \frac{v_0}{c} = \frac{v_0}{N_c} = P(x_0) + \frac{v_0}{N_c}
\]
  \[
P(x_2) = \int_{x_0}^{x_2} p(x) \, dx = \int_{x_0}^{x_1} p(x) \, dx + \int_{x_1}^{x_2} p(x) \, dx = P(x_1) + \frac{v_1}{N_c}
\]
  \[
P(x_i) = P(x_{i-1}) + \frac{v_{i-1}}{N_c}
\]
- CDF is linear between \( x_i \) and \( x_{i+1} \) with slope \( \frac{v_i}{c} \)
- Sample generation
  - Generate uniform random samples \( 0 \leq \xi \leq 1 \)
  - Compute \( x_i \) with \( P(x_i) \leq \xi \) and \( \xi < P(x_{i+1}) \)
  - Compute \( d \) with \( P(x_i) + d(P(x_{i+1}) - P(x_i)) = \xi \)
  - \( X = x_i + d(x_{i+1} - x_i) = x_i + \frac{d}{N} \) are samples from \( p(x) = \frac{1}{c} f(x) \)
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Rejection Method

- Draws samples according to a function $f(x)$
  - Dart-throwing approach
  - Works with a PDF $p(x)$ and a scalar $c$ with $f(x) < c \cdot p(x)$

- Properties
  - $f(x)$ is not necessarily a PDF
  - PDF, CDF and inverse CDF do not have to be computed
  - Simple to implement
  - Useful for debugging purposes

[Pharr, Humphreys]
Rejection Method

- Sample generation
  - Generate a uniform random sample $0 \leq \xi < 1$
  - Generate a sample $X$ according to $p(x)$
  - Accept $X$ if $\xi \cdot c \cdot p(X) \leq f(X)$

[Pharr, Humphreys]
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Transforming Between Distributions

- Computation of a resulting PDF, when a function is applied to samples from an arbitrary distribution
  - Random variables $X_i$ are drawn from $p_x(x)$
  - Bijective transformation (one-to-one mapping) $Y_i = y(X_i)$
  - How does the distribution $p_y(y)$ look like?
Transforming Between Distributions

- \( Pr\{Y \leq y(x)\} = Pr\{X \leq x\} \)

\[
P_y(y) = P_y(y(x)) = P_x(x)
\]

\[
p_y(y) = \frac{p_x(x)}{|y'(x)|}
\]

- Example \( p_x(x) = 2x \ 0 \leq x \leq 1 \)
  - \( y(x) = \sin x \ x(y) = \arcsin y \)
  - \( y'(x) = \cos x \)
  - \( p_y(y) = \frac{p_x(x)}{|\cos x|} = \frac{2x}{|\cos x|} = \frac{2 \arcsin y}{|\cos \arcsin(y)|} = \frac{2 \arcsin y}{\sqrt{1-y^2}} \)
Transforming Between Distributions

- Multiple dimensions
  - $X_i$ is an n-dimensional random variable
  - $Y_i = T(X_i)$ is a bijective transformation

- Transformation of the PDF

\[
p_y(y) = \frac{p_x(x)}{|J_T(x)|} \quad J_T(x) = \begin{pmatrix} \frac{\partial T_1}{\partial x_1} & \cdots & \frac{\partial T_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial T_n}{\partial x_1} & \cdots & \frac{\partial T_n}{\partial x_n} \end{pmatrix}
\]
Transforming Between Distributions

- Example (polar coordinates)
  - Samples \((r, \theta)\) with density \(p(r, \theta)\)
  - Corresponding density \(p(x, y)\) with \(x = r \cos \theta\) and \(y = r \sin \theta\)

\[ J_T(x) = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \quad |J_T(x)| = r(\cos^2 \theta + \sin^2 \theta) = r \]

- \(p(x, y) = \frac{1}{r} p(r, \theta)\) \(p(r, \theta) = r \cdot p(x, y)\)
Transforming Between Distributions

- Example (spherical coordinates)
  - $x = r \sin \theta \cos \phi$
  - $y = r \sin \theta \sin \phi$
  - $z = r \cos \theta$
  - $p(r, \theta, \phi) = r^2 \sin \theta \cdot p(x, y, z)$

- Example (solid angle)
  - $Pr\{\omega \in \Omega\} = \int_{\Omega} p(\omega) d\omega$
  - $d\omega = \sin \theta \, d\theta \, d\phi$
  - $p(\theta, \phi) = \sin \theta \cdot p(\omega)$
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Concept

- Samples from a 2D joint density function $p(x, y)$
- General case
  - Compute the marginal density function $p_x(x) = \int p(x, y) \, dy$
  - Compute the conditional density function $p_y(y|x) = \frac{p(x,y)}{p_x(x)}$
  - Generate a sample $X$ according to $p_x(x)$
  - Generate a sample $Y$ according to $p_y(y|X) = \frac{p(x,y)}{p_x(X)}$
- Marginal density function
  - Integral of $p(x, y)$ for a particular $x$ over all $y$-values
- Conditional density function
  - Density function for $y$ given a particular $x$
Outline

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Uniform Sampling of a Hemisphere

- PDF is constant with respect to a solid angle $p(\omega) = c$
  - $\int_{2\pi} p(\omega) d\omega = 1 \Rightarrow c \int_{2\pi} d\omega = 1 \Rightarrow c = \frac{1}{2\pi}$
  - $p(\omega) = \frac{1}{2\pi} \Rightarrow p(\theta, \phi) = \frac{\sin \theta}{2\pi}$

- Marginal density function
  - $p_{\theta}(\theta) = \int_{0}^{2\pi} p(\theta, \phi) d\phi = \int_{0}^{2\pi} \frac{\sin \theta}{2\pi} d\phi = \sin \theta$

- Conditional density for $\phi$
  - $p_{\phi}(\phi|\theta) = \frac{p(\theta, \phi)}{p_{\theta}(\theta)} = \frac{1}{2\pi}$

- Inversion method
  - $P_{\theta}(\theta) = \int_{0}^{\theta} \sin \theta' d\theta' = -\cos \theta + 1$
  - $P_{\phi}(\phi|\theta) = \int_{0}^{\phi} \frac{1}{2\pi} d\phi' = \frac{\phi}{2\pi}$
Uniform Sampling of a Hemisphere

- Inversion method cont.
  - Inverse functions of the cumulative distribution functions
  - \( \theta = \arccos(1 - \xi_1) \)
  - \( \phi = 2\pi \xi_2 \)
  - Generating uniformly sampled random values \( \xi_1 \) and \( \xi_2 \)
  - Applying the inverse CDFs to obtain \( \theta \) and \( \phi \)
- Conversion to Cartesian space
  - \( x = \sin \theta \cos \phi = \cos(2\pi \xi_2) \sqrt{1 - (1 - \xi_1)^2} \)
  - \( y = \sin \theta \sin \phi = \sin(2\pi \xi_2) \sqrt{1 - (1 - \xi_1)^2} \)
  - \( z = \cos \theta = 1 - \xi_1 \)
- \((x, y, z)^T\) is a normalized direction
Uniform Sampling of a Hemisphere

Illustration for $\theta$

$\theta = \arccos(1 - \xi_1)$

Generate less samples for smaller angles $\theta$
Uniform Sampling of a Unit Disk

- PDF is constant with respect to area $p(x, y) = \frac{1}{\pi}$
- $p(r, \theta) = r \cdot p(x, y) \Rightarrow \frac{r}{\pi}$
- Marginal density function
  - $p_r(r) = \int_0^{2\pi} p(r, \theta) d\theta = 2r$
- Conditional density
  - $p_\theta(\theta|r) = \frac{p(r, \theta)}{p_r(r)} = \frac{1}{2\pi}$
- Inversion method
  - $P_r(r) = \int_0^r 2r'dr' = r^2$
  - $P_\theta(\theta|r) = \int_0^{2\pi} \frac{1}{2\pi} d\theta' = \frac{\theta}{2\pi}$
Uniform Sampling of a Unit Disk

- Inversion method cont.
  - Inverse functions of the cumulative distribution functions
    - $r = \sqrt{\xi_1}$
    - $\theta = 2\pi \xi_2$
  - Generating uniformly sampled random values $\xi_1$ and $\xi_2$
  - Applying the inverse CDFs to obtain $r$ and $\theta$

![](image.png)

generate less samples for smaller radii
Uniform Sampling of a Cosine-Weighted Hemisphere

- PDF is proportional to \( \cos \theta \): \( p(\omega) \propto \cos \theta \)
  \[
  \int_{2\pi} \, c \, p(\omega) \, d\omega = 1 = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} c \, \cos \theta \sin \theta \, d\theta \, d\phi = c \, 2\pi \int_0^{\frac{\pi}{2}} \cos \theta \sin \theta \, d\theta = c \, 2\pi \frac{1}{2} = 1
  \]
  \( p(\theta, \phi) = \frac{1}{\pi} \cos \theta \sin \theta \)

- Marginal density function
  \( p_\theta(\theta) = \int_0^{2\pi} p(\theta, \phi) \, d\phi = \int_0^{2\pi} \frac{1}{\pi} \cos \theta \sin \theta \, d\phi = 2 \cos \theta \sin \theta \)

- Conditional density for \( \phi \)
  \( p_\phi(\phi|\theta) = \frac{p(\theta, \phi)}{p_\theta(\theta)} = \frac{1}{2\pi} \)

- Inversion method
  \[
  P_\theta(\theta) = \int_0^\theta 2 \cos \theta' \sin \theta' \, d\theta' = 2 \left[ -\frac{\cos^2 \theta'}{2} \right]_0^\theta = 2 \left( -\frac{\cos^2 \theta}{2} + \frac{1}{2} \right) = \sin^2 \theta
  \]

- \( P_\phi(\phi|\theta) = \int_0^\phi \frac{1}{2\pi} \, d\phi' = \frac{\phi}{2\pi} \)

[Suffern]
Uniform Sampling of a Cosine-Weighted Hemisphere

- Inversion method cont.
  - Inverse functions of the cumulative distribution functions
    - $\theta = \arcsin(\sqrt{\xi_1})$
    - $\phi = 2\pi \xi_2$
  - Generating uniformly sampled random values $\xi_1$ and $\xi_2$
  - Applying the inverse CDFs to obtain $\theta$ and $\phi$

- Conversion to Cartesian space
  - $x = \sin \theta \cos \phi = \cos(2\pi \xi_2) \sqrt{\xi_1}$
  - $y = \sin \theta \sin \phi = \sin(2\pi \xi_2) \sqrt{\xi_1}$
  - $z = \cos \theta = \sqrt{1 - \xi_1}$
  - $(x, y, z)^T$ is a normalized direction

x- y- values uniformly sample a unit disk, i.e., cosine-weighted samples of the hemisphere can also be obtained by uniformly sampling a unit sphere and projecting the samples onto the hemisphere.
Uniform Sampling of a Cosine-Weighted Hemisphere

Illustration for $\theta$

$\theta = \arcsin(\xi_1)$

Generate less samples for smaller and larger angles $\theta$

Cosine-weighted hemisphere (top view, side view)

Uniform hemisphere (top view)

[Suffern]
Uniform Sampling of a Triangle

- Sampling an isosceles right triangle of area 0.5
  - $u, v$ can be interpreted as Barycentric coordinates
  - Can be used to generate samples for arbitrary triangles
- $p(u, v) = 2$

- Marginal density function
  - $p_u(u) = \int_0^{1-u} p(u, v) \, dv = 2 \int_0^{1-u} dv = 2(1 - u)$

- Conditional density
  - $p_v(v|u) = \frac{p(u,v)}{p_u(u)} = \frac{1}{1-u}$

- Inversion method
  - $P_u(u) = \int_0^u 2 - 2u' \, du' = 2u - u^2$
  - $P_v(v|u) = \int_0^v \frac{1}{1-u} \, dv' = \frac{v}{1-u}$
Uniform Sampling of a Triangle

- Inversion method cont.
  - Inverse functions of the cumulative distribution functions
    - \( u = 1 - \sqrt{\xi_1} \) u is generated between 0 and 1
    - \( v = \xi_2 \sqrt{\xi_1} \) v is generated between 0 and \( 1-u=\xi_2^{1/2} \)
  - Generating uniformly sampled random values \( \xi_1 \) and \( \xi_2 \)
  - Applying the inverse CDFs to obtain \( u \) and \( v \)
Piecewise-Constant 2D Distribution

- \( n_u \times n_v \) samples defined over \((u, v) \in [0, 1]^2\)
  - E.g., an environment map
- \( f(u, v) \) is defined by a set of \( n_u \times n_v \) values \( f[u_i, v_i] \)
  - \( u_i \in [0, \ldots, n_u - 1] \quad \forall i \in [0, \ldots, n_v - 1] \)
  - \( f[u_i, v_i] \) is the value of \( f(u, v) \) in the range \([\frac{i}{n_u}, \frac{i+1}{n_u}] \times [\frac{j}{n_v}, \frac{j+1}{n_v}]\)
  - \( f(u, v) = f[u_i, v_i] \) with \( \tilde{u} = [n_u u] \) and \( \tilde{v} = [n_v v] \)
- Integral over the domain
  - \( I_f = \int \int f(u, v) \, du \, dv = \frac{1}{n_u n_v} \sum_i \sum_j f[u_i, v_j] \)
- PDF
  - \( p(u, v) = \frac{1}{I_f} f(u, v) = \frac{1}{I_f} f[\tilde{u}, \tilde{v}] \)
Piecewise-Constant 2D Distribution

- Marginal density function
  \[ p_v(v) = \int p(u, v) \, du = \frac{1}{I_f} \frac{1}{n_u} \sum_i f[u_i, \tilde{v}] \]
- Piecewise-constant 1D function
- Defined by \( n_v \) values \( p_v[\tilde{v}] \)

- Conditional density
  \[ p_u(u|v) = \frac{p(u,v)}{p_v(v)} = \frac{1}{I_f} \frac{f[\tilde{u}, \tilde{v}]}{p[\tilde{v}]} \]
- Piecewise-constant 1D function

- Sample generation
  - See example 3 of the inversion method
Piecewise-Constant 2D Distribution

- Environment map
- Low-resolution of the marginal density function and the conditional distributions for rows
- First, a row is selected according to the marginal density function
- Then, a column is selected from the row's 1D conditional distribution