Advanced Computer Graphics
Sampling Strategies for Solving the Rendering Equation

Matthias Teschner

UNI FREIBURG

Outline

- Context
- Some concepts
- Direct illumination
- Indirect illumination

Goal and Governing Equation

- Computation of incident radiance at a sensor $L(\boldsymbol{s} \leftarrow \boldsymbol{\omega}_s)$
- Incident radiance at sensor position $\boldsymbol{s}$ is equal to exitant radiance at scene position $\boldsymbol{p}$ with $\boldsymbol{p} = \boldsymbol{r}_c(\boldsymbol{s}, \boldsymbol{\omega}_s)$:

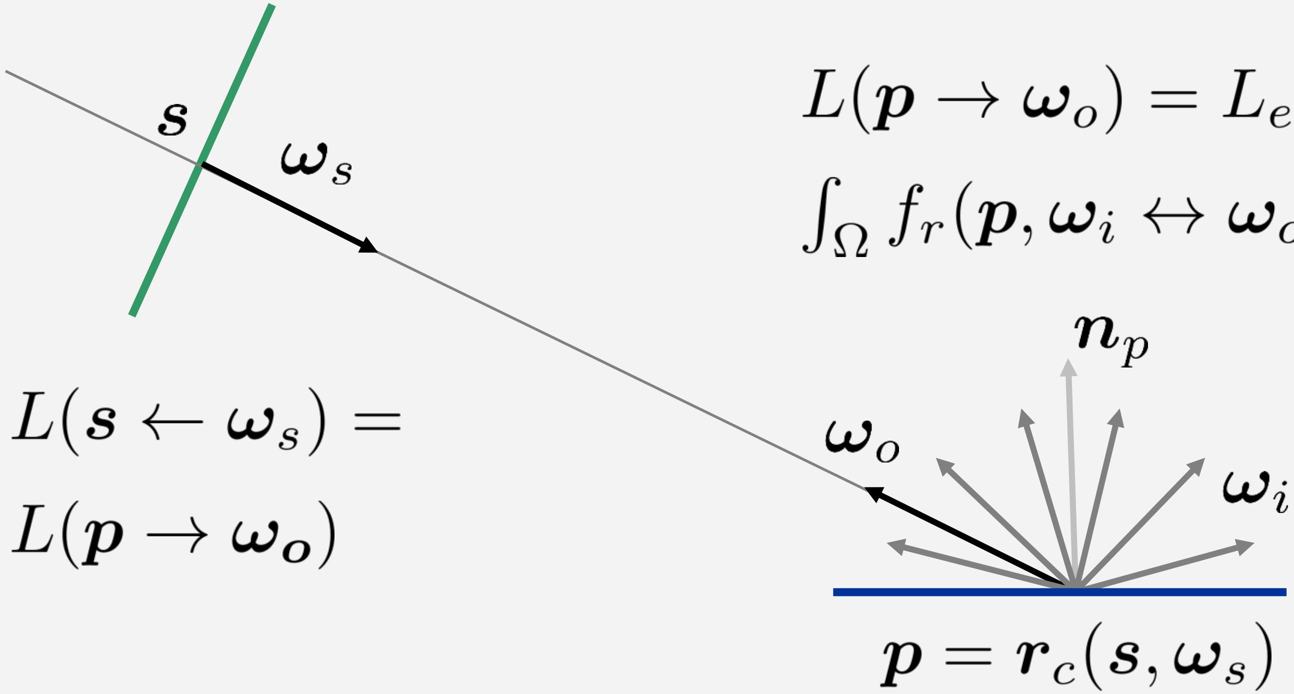
$$L(\boldsymbol{s} \leftarrow \boldsymbol{\omega}_s) = L(\boldsymbol{p} \rightarrow \boldsymbol{\omega_o})$$

  - Raycast operator $\boldsymbol{r}_c$, conservation of radiance
- Exitant radiance at scene position $\boldsymbol{p}$ is computed as:

$$L(\boldsymbol{p} \rightarrow \boldsymbol{\omega}_o) = L_e(\boldsymbol{p} \rightarrow \boldsymbol{\omega}_o) + \int_\Omega f_r(\boldsymbol{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L(\boldsymbol{p} \leftarrow \boldsymbol{\omega}_i) \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p) \mathrm{d}\omega_i$$

  - Rendering equation

Goal and Governing Equation



$$L(\boldsymbol{s} \leftarrow \boldsymbol{\omega}_s) = L(\boldsymbol{p} \rightarrow \boldsymbol{\omega_o})$$

$$L(\boldsymbol{p} \rightarrow \boldsymbol{\omega}_o) = L_e(\boldsymbol{p} \rightarrow \boldsymbol{\omega}_o) + \int_\Omega f_r(\boldsymbol{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L(\boldsymbol{p} \leftarrow \boldsymbol{\omega}_i) \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p) \mathrm{d}\omega_i$$

Monte Carlo Integration

$$L(\boldsymbol{p} \rightarrow \boldsymbol{\omega}_o) = L_e(\boldsymbol{p} \rightarrow \boldsymbol{\omega}_o) + \int_\Omega f_r(\boldsymbol{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L(\boldsymbol{p} \leftarrow \boldsymbol{\omega}_i) \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p) \mathrm{d}\omega_i$$

is approximated with

$$L(\boldsymbol{p} \rightarrow \boldsymbol{\omega}_o) = L_e(\boldsymbol{p} \rightarrow \boldsymbol{\omega}_o) + \sum_{i=1}^N f_r(\boldsymbol{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L(\boldsymbol{p} \leftarrow \boldsymbol{\omega}_i) \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p) \frac{1}{N p(\boldsymbol{\omega_i})}$$

- N randomly sampled directions $\boldsymbol{\omega}_i$
- According to a probability density function $p(\boldsymbol{\omega}_i)$

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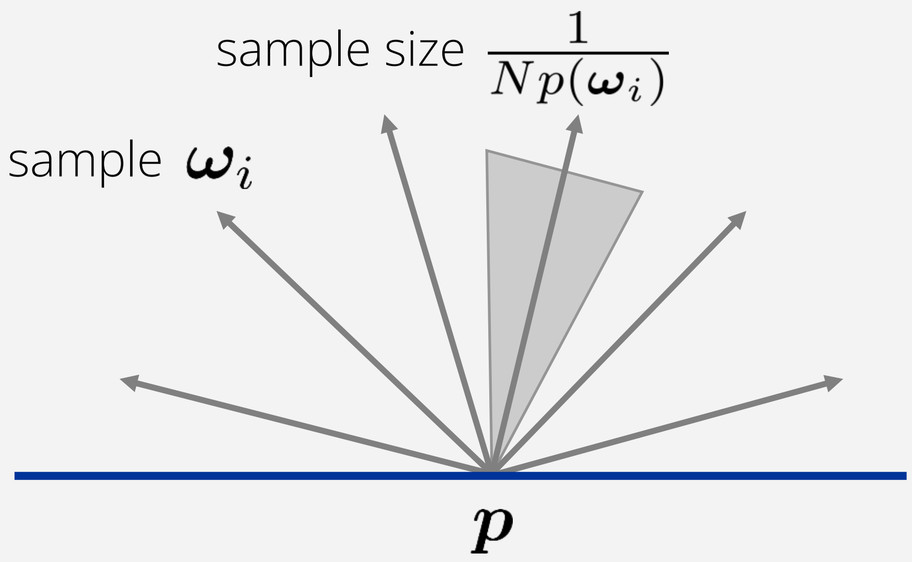
Monte Carlo Integration

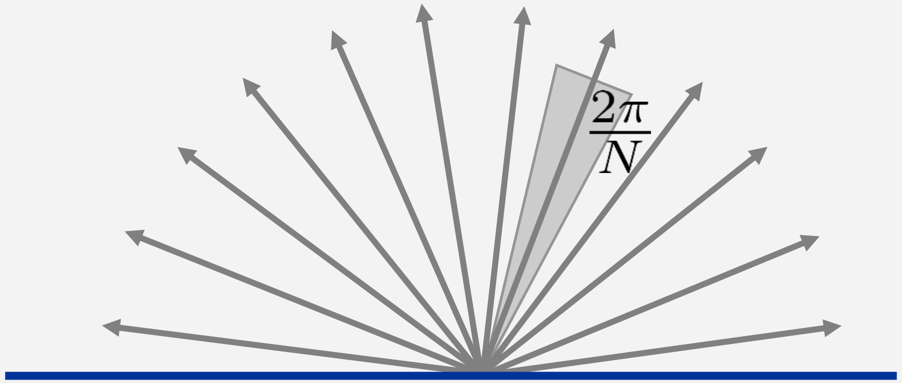
$$\int_\Omega f_r(\boldsymbol{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L(\boldsymbol{p} \leftarrow \boldsymbol{\omega}_i) \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p) \mathrm{d}\omega_i$$

$$\approx \sum_{i=1}^{N} f_r(\boldsymbol{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L(\boldsymbol{p} \leftarrow \boldsymbol{\omega}_i) \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p) \frac{1}{N p(\boldsymbol{\omega_i})}$$

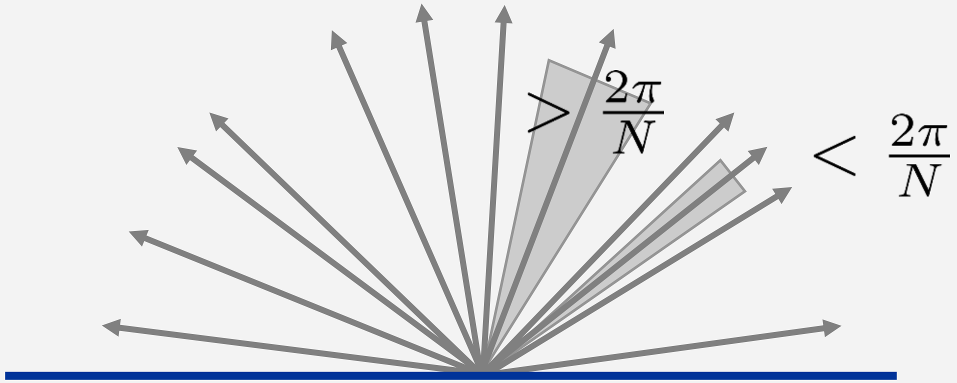


Monte Carlo Integration – Error

- Estimated sample size is not equal to the actual sample size due to random sample selection
- Sample contributions are randomly over- or underestimated



Uniform sampling of a 3D hemisphere
Estimated sample size



Uniform random sampling of a 3D hemisphere
Actual sample size

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Monte Carlo Integration - Error

- Variance, noise: resulting radiance values are randomly too dark or too bright
- If a Monte Carlo approximation converges for growing sample numbers to the correct result, the scheme is unbiased, otherwise biased

Monte Carlo Integration - Variance

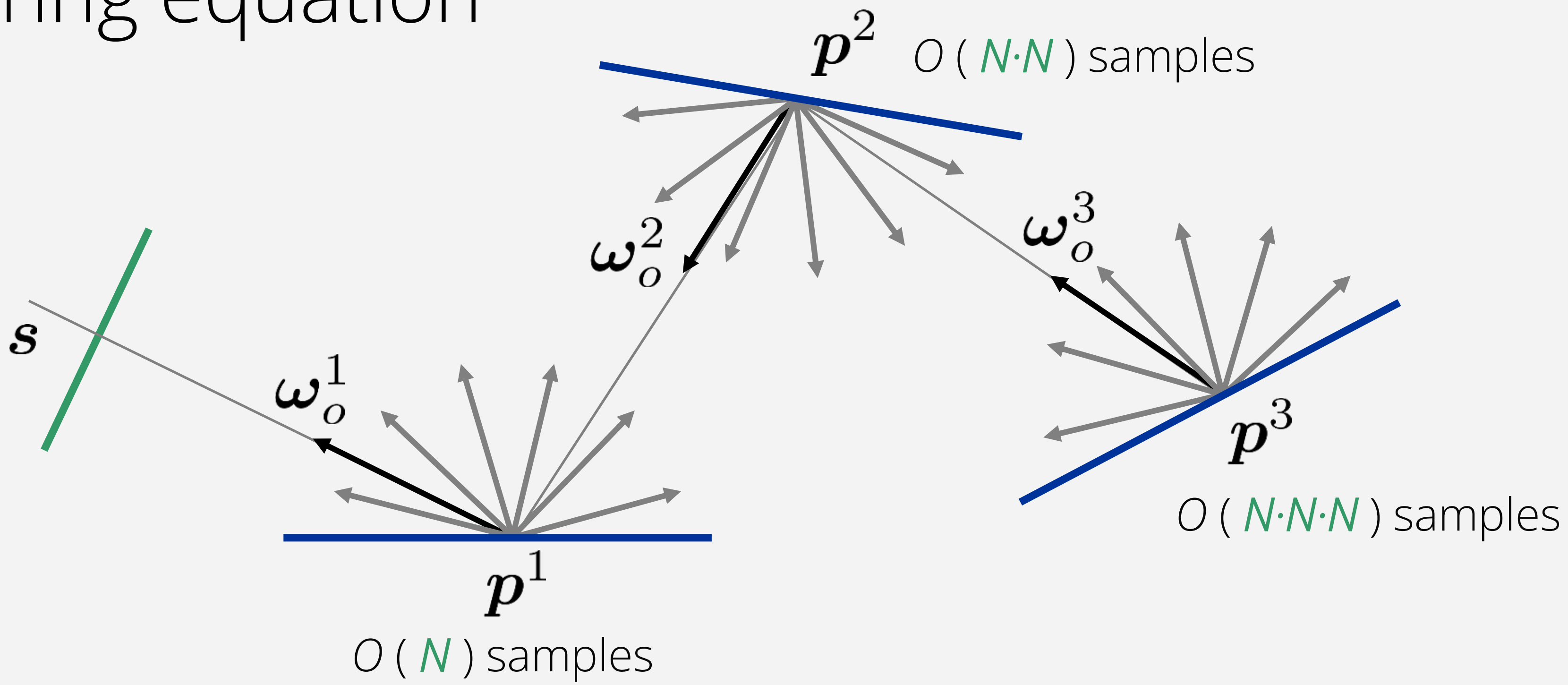
8 samples per pixel

1024 samples per pixel

[Pharr et al., Physically Based Rendering]

Rendering Equation – Recursive Problem

– Solving the rendering equation requires *N* samples, where many samples require the solution of another rendering equation



Need of a Sampling Strategy

- Sample processing is expensive
  - Ray-scene intersection tests
- Samples differ in terms of relevance
- Important samples, e.g.
  - Towards / from visible light sources
  - From / towards sensors
  - Towards / from bright parts of a scene
- Less important samples, e.g.
  - After increasing number of bounces
  - Towards / from dark parts in a scene

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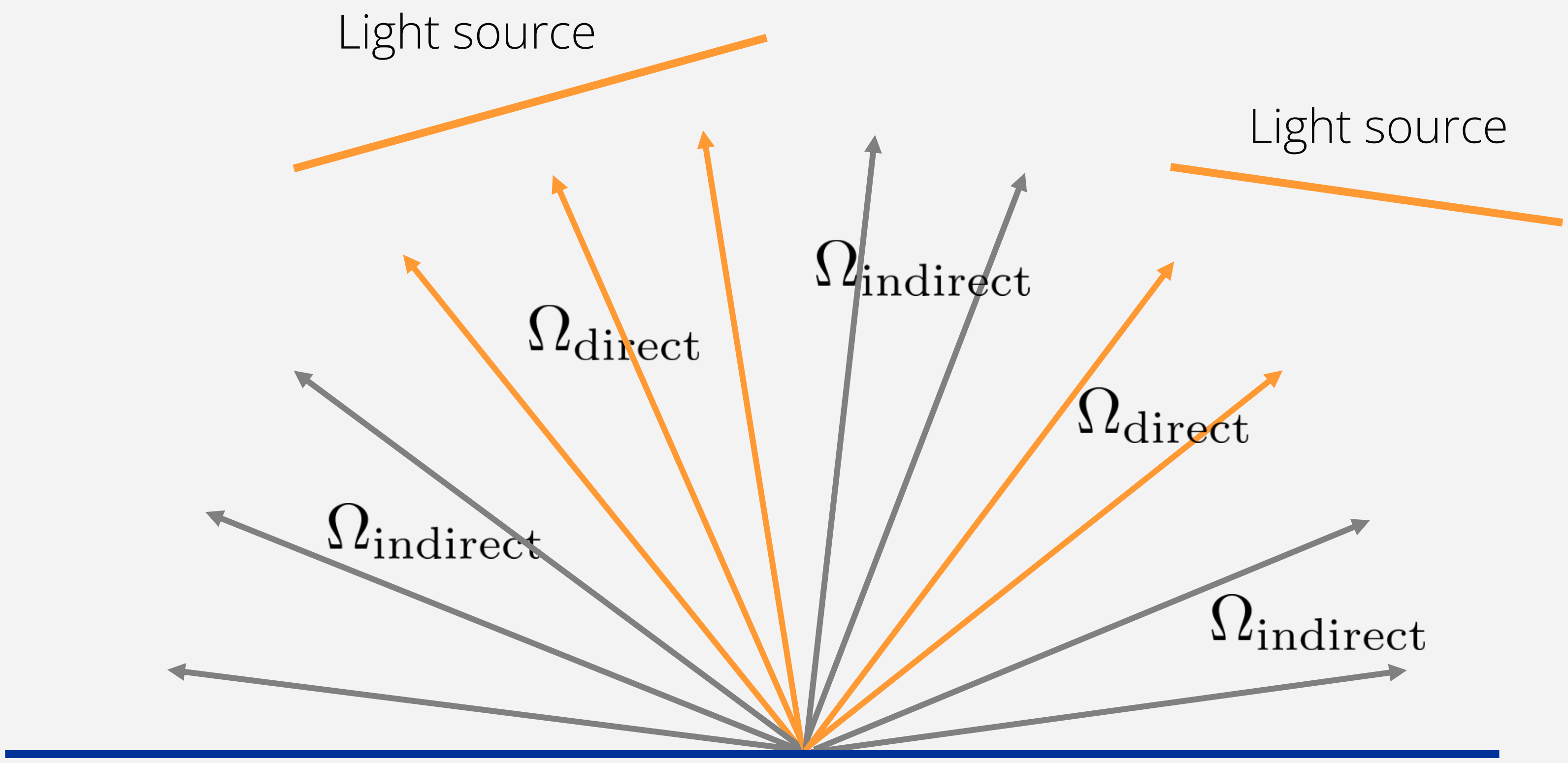
Stratification

- Subdivision of the integration domain, e.g.

$$L(\boldsymbol{p} \to \boldsymbol{\omega}_o) = L_e(\boldsymbol{p} \to \boldsymbol{\omega}_o) + \int_\Omega f_r(\boldsymbol{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L(\boldsymbol{p} \leftarrow \boldsymbol{\omega}_i) \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p) \mathrm{d}\omega_i$$
$$= L_e(\boldsymbol{p} \to \boldsymbol{\omega}_o) + \int_{\Omega_\text{direct}} f_r(\boldsymbol{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L_e(\boldsymbol{p} \leftarrow \boldsymbol{\omega}_i) \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p) \mathrm{d}\omega_i$$
$$+ \int_{\Omega_\text{indirect}} f_r(\boldsymbol{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L(\boldsymbol{p} \leftarrow \boldsymbol{\omega}_i) \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p) \mathrm{d}\omega_i$$

- Integral over Ω_direct can directly be computed using L_e
- Integral over Ω_indirect requires the recursive computation of L

Stratification



Two sample sets for two parts of the integration domain

Stratification

- Subdivision into non-overlapping strata
  - Allows the usage of an individual technique for each stratum
  - Allows / requires the individual sampling of each stratum
  - Avoids sample clustering in a part of the integration domain

Importance Sampling

- Probability density function
  - Should be proportional to the integrand

    $$p(\boldsymbol{\omega}_i) \propto f_r(\boldsymbol{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L(\boldsymbol{p} \leftarrow \boldsymbol{\omega}_i) \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p)$$

    - Product of functions
    - Incident radiance expensive to compute
  - Optimal PDF

    $$p(\boldsymbol{\omega}_i) = \frac{f_r(\boldsymbol{p},\boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L(\boldsymbol{p} \leftarrow \boldsymbol{\omega}_i) \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p)}{\int_\Omega f_r(\boldsymbol{p},\boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L(\boldsymbol{p} \leftarrow \boldsymbol{\omega}_i) \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p) \mathrm{d}\omega_i}$$

    - Irrelevant: If the integral would be known, we are done.

Importance Sampling

- Large integrand values
  - More samples with smaller size and reduced sampling inaccuracies to improve accuracy, i.e. minimize variance / noise
- Small integrand values
  - Less samples with larger size and larger sampling errors to improve efficiency

Multiple Importance Sampling MIS

– Combine sample sets from different PDFs

$$\int_\Omega f(x)\mathrm{d}x \approx \sum_{j=1}^{M} f(X_j)\frac{1}{Mp(X_j)}$$

Monte Carlo with M samples

$$= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{M} f(X_j)\frac{1}{Mp(X_j)}$$

Summing up N MC estimates and dividing by N

$$= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N_i} f(X_{i,j})\frac{1}{N_i p_i(X_{i,j})}$$

Using individual PDFs p_i with individual sample counts N_i for each of the N MC estimates

$$= \sum_{j=1}^{N_1}\frac{1}{N} f(X_{1,j})\frac{1}{N_1 p_1(X_{1,j})} + \sum_{j=1}^{N_2}\frac{1}{N} f(X_{2,j})\frac{1}{N_2 p_2(X_{2,j})} + \ldots + \sum_{j=1}^{N_N}\frac{1}{N} f(X_{N,j})\frac{1}{N_N p_N(X_{N,j})}$$

Replacing weight $1/N$ with individual weighting functions w_i

$$= \sum_{j=1}^{N_1} w_1(X_{1,j}) f(X_{1,j})\frac{1}{N_1 p_1(X_{1,j})} + \sum_{j=1}^{N_2} w_2(X_{2,j}) f(X_{2,j})\frac{1}{N_2 p_2(X_{2,j})} + \ldots$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N_i} w_i(X_{i,j}) f(X_{i,j})\frac{1}{N_i p_i(X_{i,j})} = \sum_{i=1}^{N}\frac{1}{N_i}\sum_{j=1}^{N_i} w_i(X_{i,j})\frac{f(X_{i,j})}{p_i(X_{i,j})}$$

Multiple Importance Sampling MIS

$$\int_\Omega f(x)\mathrm{d}x \approx \sum_{i=1}^{N} \frac{1}{N_i} \sum_{j=1}^{N_i} w_i(X_{i,j})) \frac{f(X_{i,j})}{p_i(X_{i,j})}$$

- Use N PDFs p_i
- Generate N_i samples $X_{i,j}$ from PDF p_i
- Weight all contributions with functions $w_i(x) : \Omega \to \mathbb{R}$
  - Constraints for weighting functions

$$f(x) \neq 0 \Rightarrow \sum_i w_i(x) = 1$$

The weights have to add up to one everywhere on Ω. The weights are irrelevant, if a sample has zero contribution.

$$p_j(x) = 0 \Rightarrow w_j(x) = 0$$

$$\Rightarrow \sum_{i \neq j} w_i(x) = 1$$

If any of the PDFs is zero for some x, the weights for all other PDFs have to sum up to one.

MIS – Example Weightings

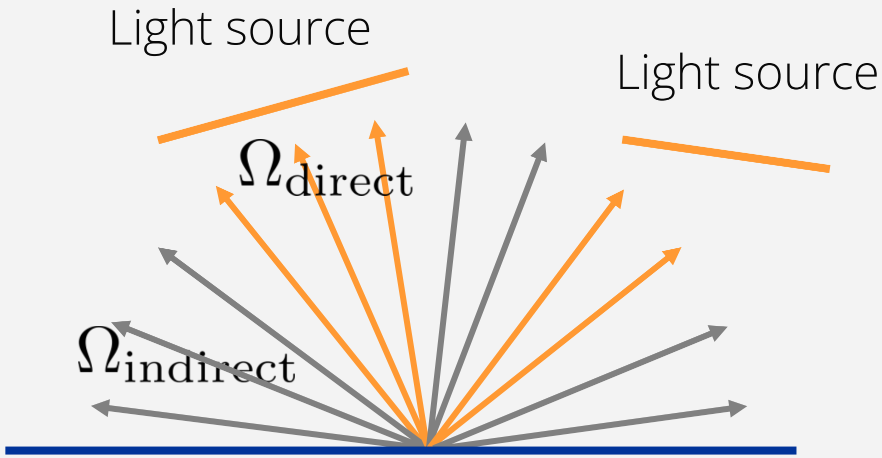
$$\int_\Omega f(x)\mathrm{d}x \approx \sum_{i=1}^{N} \frac{1}{N_i} \sum_{j=1}^{N_i} w_i(X_{i,j})) \frac{f(X_{i,j})}{p_i(X_{i,j})}$$

$$w_i(x)) \in \{0,1\} \qquad \sum_i w_i(x) = 1$$

$$X_{i,j} \in \Omega_{\text{indirect}} \Rightarrow w_1(X_{i,j}) = 1 \wedge w_2(X_{i,j}) = 0$$

$$X_{i,j} \in \Omega_{\text{direct}} \Rightarrow w_1(X_{i,j}) = 0 \wedge w_2(X_{i,j}) = 1$$

- Realizes stratification
- E.g. generate samples from p_1 and p_2
- Use a sample from p_1, if it is in Ω_{indirect} and discard it if it is in Ω_{direct}
- Use a sample from p_2, if it is in Ω_{direct} and discard it if it is in Ω_{indirect}



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MIS – Example Weightings

$$\int_\Omega f(x)\mathrm{d}x \approx \sum_{i=1}^{N} \frac{1}{N_i} \sum_{j=1}^{N_i} w_i(X_{i,j})) \frac{f(X_{i,j})}{p_i(X_{i,j})}$$

$$w_i(X_{i,j}) = \frac{1}{N}$$

- Compute N MC estimates with PDFs p_i and average them

$$w_i(X_{i,j})) = \frac{p_i(X_{i,j})}{\sum_{k=1}^{N} p_k(X_{i,j})}$$

- Balance heuristic [Eric Veach 1995, 1997]
- Larger weight to more accurate samples with smaller size
- Good, if any of the p_i is large for large f, but no p_i is proportional to f everywhere
- If any p_i is perfectly proportional to f, the balance heuristic is not optimal

MIS – Example Weightings

$$\int_\Omega f(x)\mathrm{d}x \approx \sum_{i=1}^{N} \frac{1}{N_i} \sum_{j=1}^{N_i} w_i(X_{i,j})) \frac{f(X_{i,j})}{p_i(X_{i,j})}$$

$$w_i(X_{i,j})) = \frac{p_i(X_{i,j})^\beta}{\sum_{k=1}^{N} p_k(X_{i,j})^\beta} \qquad \beta = 2$$

- Power heuristic [Eric Veach 1995, 1997]
- Popular choice in MIS

– Other alternatives

- Cutoff heuristic
- Maximum heuristic

MIS – Adaptive Sample Counts

$$\int_\Omega f(x)\mathrm{d}x \approx \sum_{i=1}^{N} \frac{1}{N_i} \sum_{j=1}^{N_i} w_i(X_{i,j})) \frac{f(X_{i,j})}{p_i(X_{i,j})}$$

- Fixed sample counts N_i can be replaced by randomly selecting a PDF p_i from a discrete PDF $p(i)$
- One-sample estimator
  - Generate I_k from p — Choose a PDF
  - Generate X_k from p_{I_k} — Draw a sample from that PDF

$$\int_\Omega f(x)\mathrm{d}x \approx \frac{1}{N} \sum_{k=1}^{N} \frac{w_{I_k}(X_k) f(X_k)}{p(I_k) p_{I_k}(X_k)} \approx \frac{w_{I_1}(X_{I_1}) f(X_1)}{p(I_1) p_{I_1}(X_1)}$$

  - Relevant, e.g. in path tracing

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MIS - Example

- Diffuse material under direct illumination L_e
  - Regular importance sampling with a PDF $p_1(\boldsymbol{\omega}_i) \propto \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p)$

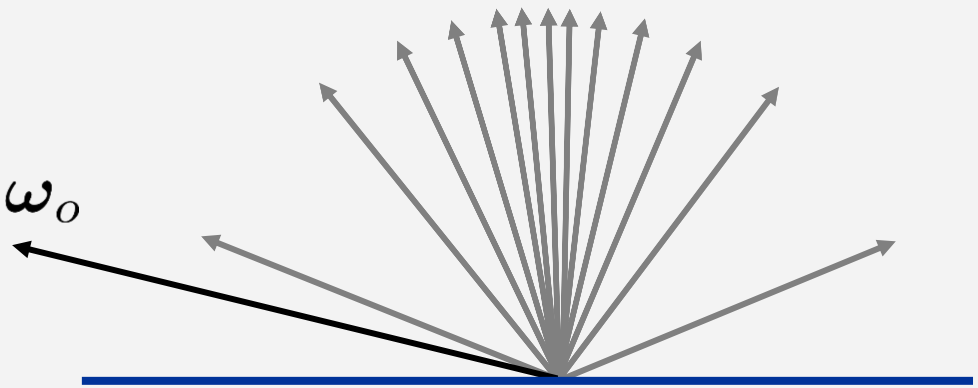
    $\int_\Omega \frac{\rho_d}{\pi} L_e \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p) \mathrm{d}\omega_i \approx \sum_{i=1}^N \frac{\rho_d}{\pi} L_e \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p) \frac{1}{N p_1(\boldsymbol{\omega}_i)}$
- Mixed material under L_e
  - Multiple importance sampling with two PDFs p_1 and p_2 with $p_1(\boldsymbol{\omega}_i) \propto \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p)$ and $p_2(\boldsymbol{\omega}_i) \propto \cos(\boldsymbol{r}(\boldsymbol{n}_p, \boldsymbol{\omega}_i), \boldsymbol{\omega}_o)^e$

    $\int_\Omega (\frac{\rho_d}{\pi} + \rho_g \cos(\boldsymbol{r}(\boldsymbol{n}_p, \boldsymbol{\omega}_i), \boldsymbol{\omega}_o)^e) L_e \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p) \mathrm{d}\omega_i$ *r* – reflection direction

    $\approx \frac{1}{N} \sum_{i=1}^N \frac{w_{I_i}(\boldsymbol{\omega}_i)}{p(I_i) p_{I_i}(\boldsymbol{\omega}_i)} \left(\frac{\rho_d}{\pi} + \rho_g \cos(\boldsymbol{r}(\boldsymbol{n}_p, \boldsymbol{\omega}_i), \boldsymbol{\omega}_o)^e\right) L_e \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p)$

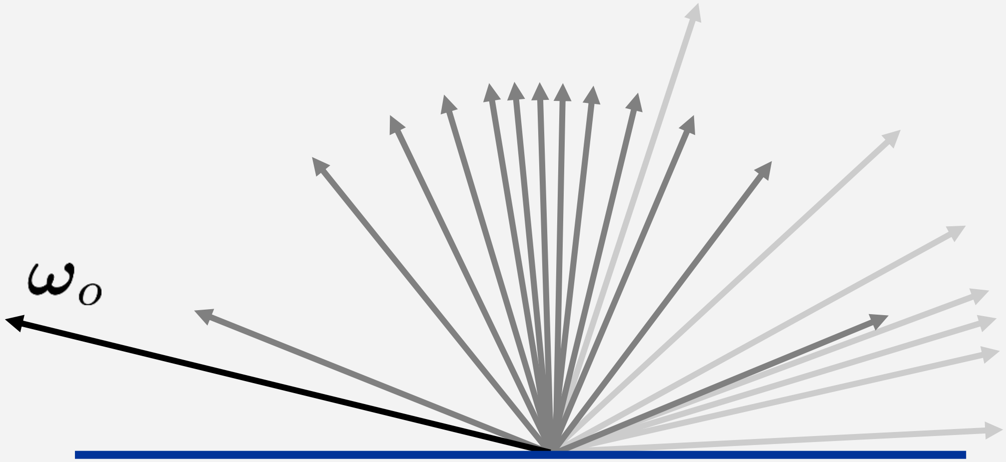
    $I_i \in \{1, 2\}$ from p, e.g. $p(1) = p(2) = \frac{1}{2}$, $\boldsymbol{\omega}_i$ from p_{I_i}

MIS - Example



Importance sampling
for a diffuse surface

Using samples from one PDF



Multiple importance sampling
for mixed material

Using two sample sets from two PDFs

Weighted averaging of two MC estimates

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Problem

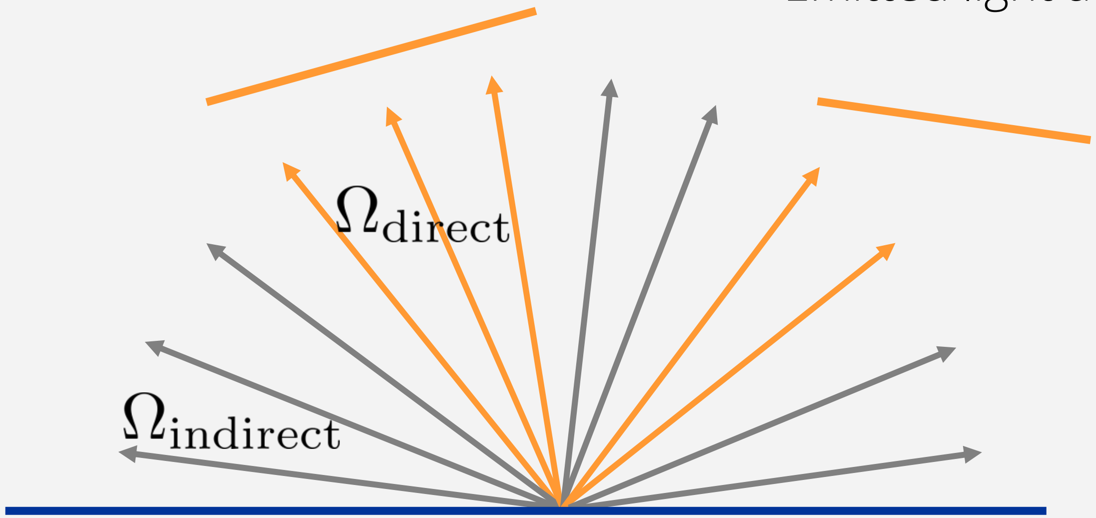
– Computation of $\int_{\Omega_{\text{direct}}}$ from the rendering equation

$$L_o(\boldsymbol{p} \to \boldsymbol{\omega}_o) = L_e(\boldsymbol{p} \to \boldsymbol{\omega}_o) + \int_{\Omega_{\text{direct}}} f_r(\boldsymbol{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L_e(\boldsymbol{p} \leftarrow \boldsymbol{\omega}_i) \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p) \mathrm{d}\omega_i$$

Emitted light Emitted light after 1 bounce

$$+ \int_{\Omega_{\text{indirect}}} f_r(\boldsymbol{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L(\boldsymbol{p} \leftarrow \boldsymbol{\omega}_i) \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p) \mathrm{d}\omega_i$$

Emitted light after *n* bounces with *n*>1



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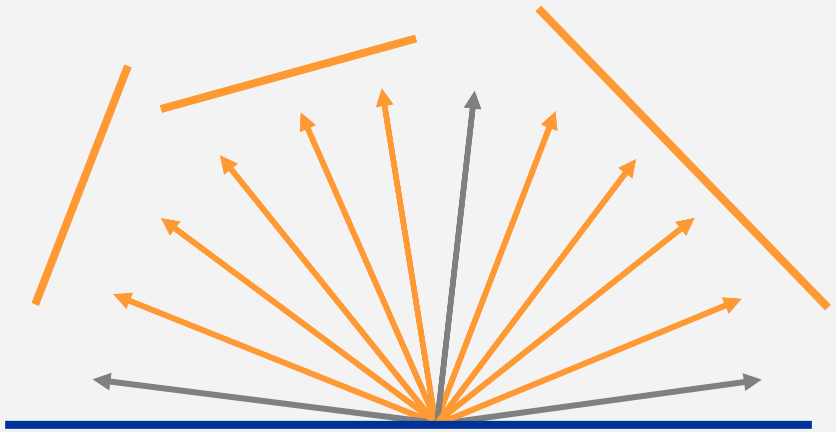
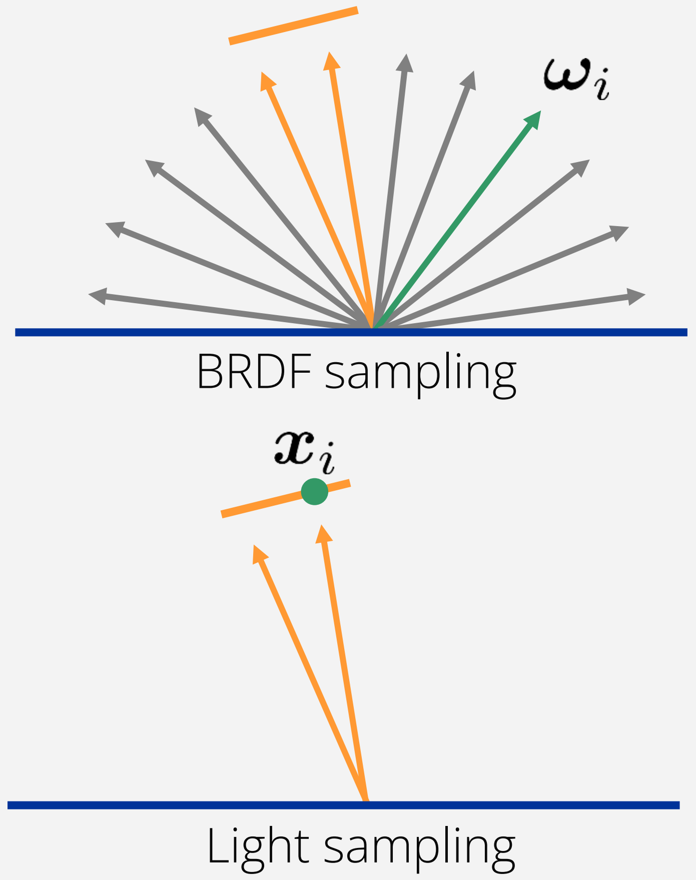
Hemisphere Dominated by L_e

- BRDF sampling
- Sampling directions from a PDF proportional to the BRDF
- Diffuse: $p_1(\boldsymbol{\omega}_i) \propto \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p)$
  $\sum_{i=1}^{N} \frac{\rho_d}{\pi} L_e \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p) \frac{1}{N p_1(\boldsymbol{\omega}_i)}$
- Mixed: $p_1(\boldsymbol{\omega}_i) \propto \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p) \quad p_2(\boldsymbol{\omega}_i) \propto \cos(\boldsymbol{r}(\boldsymbol{n}_p, \boldsymbol{\omega}_i), \boldsymbol{\omega}_o)^e$
  $\frac{1}{N} \sum_{i=1}^{N} \frac{w_{I_i}(\boldsymbol{\omega}_i)}{p(I_i) p_{I_i}(\boldsymbol{\omega}_i)} \left(\frac{\rho_d}{\pi} + \rho_g \cos(\boldsymbol{r}(\boldsymbol{n}_p, \boldsymbol{\omega}_i), \boldsymbol{\omega}_o)^e \right) L_e \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p)$
- Majority of samples hit a light source, only few misses with zero contribution



Small Light Source

- Majority of samples would miss in case of BRDF sampling, inefficient
- Light sampling
  - Use area form of the rendering equation
  - Sample positions on the light source instead of directions



$$\int_{\Omega_{\text{direct}}} f_r(\boldsymbol{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L_e(\boldsymbol{p} \leftarrow \boldsymbol{\omega}_i) \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p) \mathrm{d}\omega_i$$

$$= \int_{A_{\text{direct}}} f_r(\boldsymbol{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L_e(\boldsymbol{x} \rightarrow -\boldsymbol{\omega}_i) V(\boldsymbol{p}, \boldsymbol{x}) \frac{\cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p) \cos(-\boldsymbol{\omega}_i, \boldsymbol{n}_x)}{r_{px}^2} \mathrm{d}x$$

Light Sampling

$$\int_{A_{\text{direct}}} f_r(\boldsymbol{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L_e(\boldsymbol{x} \rightarrow -\boldsymbol{\omega}_i) V(\boldsymbol{p}, \boldsymbol{x}) \frac{\cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p) \cos(-\boldsymbol{\omega}_i, \boldsymbol{n}_x)}{r_{px}^2} \mathrm{d}x$$

$$= \int_{A_{\text{direct}}} f_r(\boldsymbol{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L_e(\boldsymbol{x} \rightarrow -\boldsymbol{\omega}_i) G(\boldsymbol{p}, \boldsymbol{x}) \mathrm{d}x$$

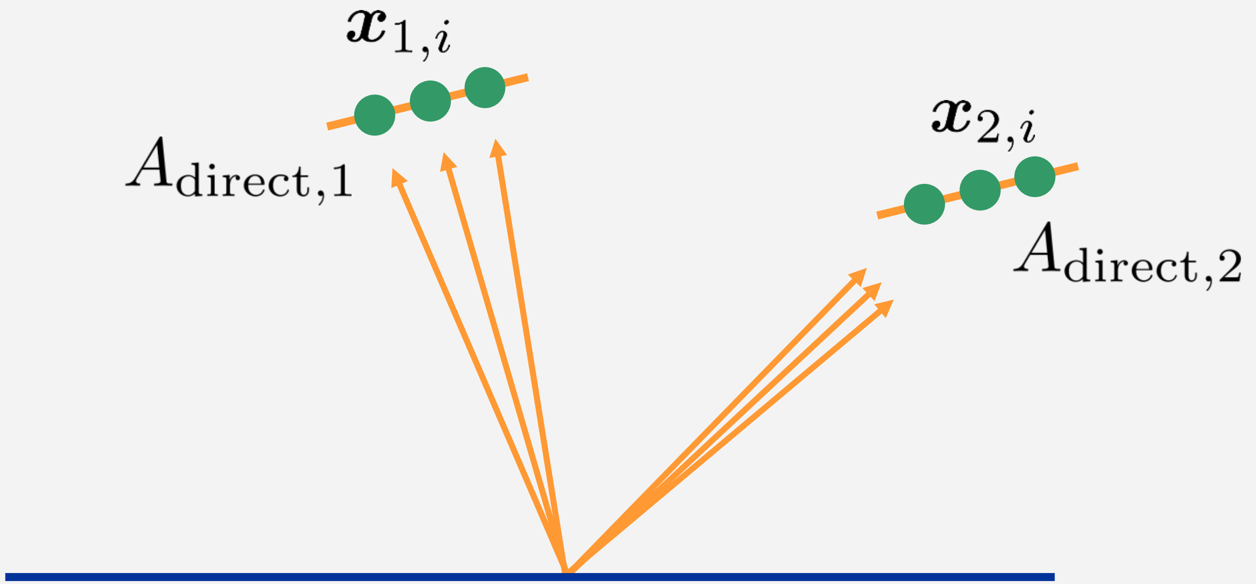
$$\approx \sum_{i=1}^{N} f_r(\boldsymbol{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L_e(\boldsymbol{x}_i \rightarrow -\boldsymbol{\omega}_i) G(\boldsymbol{p}, \boldsymbol{x}_i) \frac{1}{N p(\boldsymbol{x}_i)}$$

- E.g., uniform light sampling $p(\boldsymbol{x}_i) = \frac{1}{A_{\text{direct}}}$
- Area of the light source A_{direct}
- Position $\boldsymbol{x}_i$ is sampled, direction $\boldsymbol{\omega}_i$ is computed as $\boldsymbol{\omega}_i = \frac{\boldsymbol{x}_i - \boldsymbol{p}}{||\boldsymbol{x}_i - \boldsymbol{p}||}$

Many Small Light Sources

- N_l light sources with areas $A_{\text{direct},j}$
- Uniform sampling of all light sources

$$\sum_{j=1}^{N_l} \sum_{i=1}^{N} f_r(\boldsymbol{p}, \boldsymbol{\omega}_{j,i} \leftrightarrow \boldsymbol{\omega}_o) L_e(\boldsymbol{x}_{j,i} \rightarrow -\boldsymbol{\omega}_{j,i}) G(\boldsymbol{p}, \boldsymbol{x}_{j,i}) \frac{1}{N p_j(\boldsymbol{x}_{j,i})}$$



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Adaptive Sample Counts

- Random light source selection from a discrete PDF p
- One-sample estimator
  - Generate I_k from p — Choose a PDF
  - Generate positions $\boldsymbol{x}_k$ from p_{I_k} — Draw a sample from that PDF
  - Compute $\boldsymbol{\omega}_k = \frac{\boldsymbol{x}_k - \boldsymbol{p}}{\|\boldsymbol{x}_k - \boldsymbol{p}\|}$
  - MC estimator

    $\sum_{k=1}^{N} f_r(\boldsymbol{p}, \boldsymbol{\omega}_k \leftrightarrow \boldsymbol{\omega}_o) L_e(\boldsymbol{x}_k \rightarrow -\boldsymbol{\omega}_k) G(\boldsymbol{p}, \boldsymbol{x}_k) \frac{1}{N p(I_k) p_{I_k}(\boldsymbol{x}_k)}$

    $\approx f_r(\boldsymbol{p}, \boldsymbol{\omega}_1 \leftrightarrow \boldsymbol{\omega}_o) L_e(\boldsymbol{x}_1 \rightarrow -\boldsymbol{\omega}_1) G(\boldsymbol{p}, \boldsymbol{x}_1) \frac{1}{p(I_1) p_{I_1}(\boldsymbol{x}_1)}$
  - Relevant, e.g. in path tracing

Light Source Sampling

- Random light source selection
  - Based on relevance for $\int_{\Omega_{\mathrm{direct}}} \dots$
  - Discrete PDF p should be proportional to
    - Projected light source area
    - Light source power
- Sampling of each light source
  - Proportional to spatial power distribution

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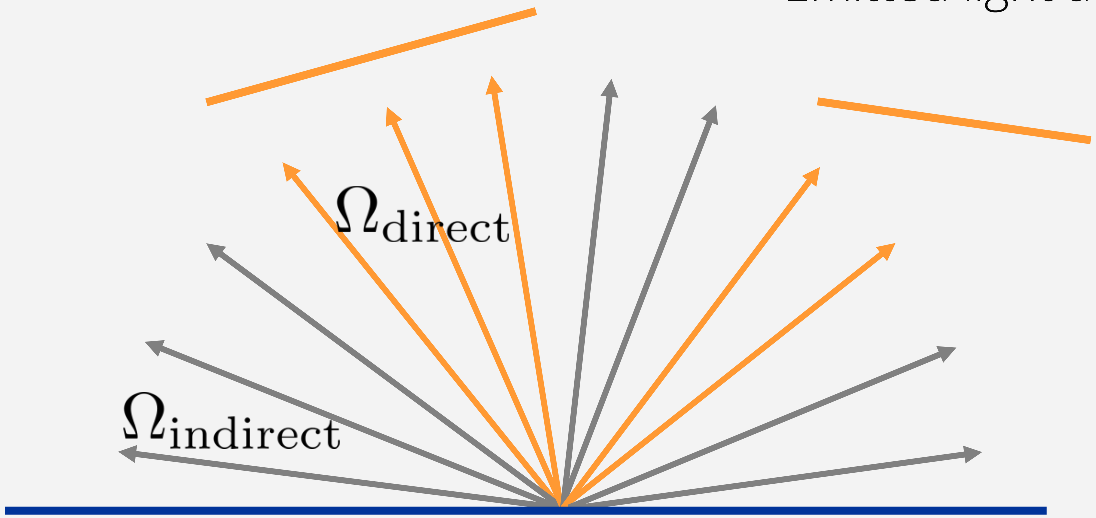
Emitted light — Emitted light after 1 bounce

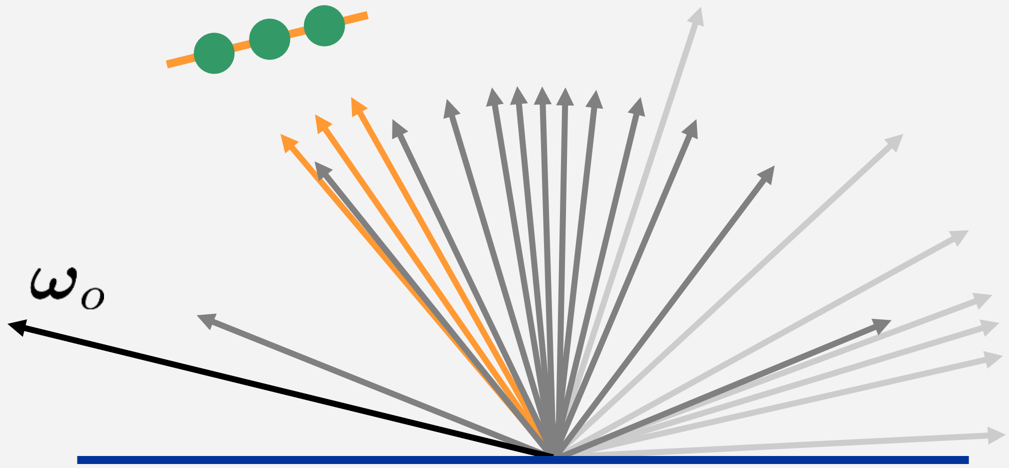
$$+ \int_{\Omega_{\text{indirect}}} f_r(\boldsymbol{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L(\boldsymbol{p} \leftarrow \boldsymbol{\omega}_i) \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p) \mathrm{d}\omega_i$$

Emitted light after *n* bounces with *n*>1



Combination of Ω_{direct} and $\Omega_{indirect}$

- $\int_{\Omega_{\mathrm{direct}}}$ can already be computed
- Assume, $\int_{\Omega_{\mathrm{indirect}}}$ can also be computed



Multiple importance sampling
for mixed material
and a small light source

Using three sample sets from three PDFs
Weighted averaging of three MC estimates
Relevant for recursive raytracing to compute
$\int_{\Omega_{\mathrm{direct}}} \dots + \int_{\Omega_{\mathrm{indirect}}} \dots$

Combination of Ω_{direct} and $\Omega_{indirect}$

- Three PDFs
  - BRDF PDFs $p_1(\boldsymbol{\omega}_i) \propto \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p)$ $p_2(\boldsymbol{\omega}_i) \propto \cos(\boldsymbol{r}(\boldsymbol{n}_p, \boldsymbol{\omega}_i), \boldsymbol{\omega}_o)^e$
  - Light PDF $p_3(\boldsymbol{x}_i) = \frac{1}{A_{\text{direct}}}$
  - Discrete PDF for PDF selection, e.g. $p(1) = p(2) = p(3) = \frac{1}{3}$

$F = 0$

Select $I_i \in \{1, 2, 3\}$ from p

Generate *N* samples

$I_i \in \{1, 2\} \wedge \boldsymbol{\omega}_i \in \Omega_{\text{indirect}} \Rightarrow$

If a sample direction from p_1 or p_2 does not hit the light source, it contributes to $\int_{\Omega_{\text{indirect}}} \dots$

$$F = F + \frac{w_{I_i}(\boldsymbol{\omega}_i)}{p(I_i) p_{I_i}(\boldsymbol{\omega}_i)} f_r(\boldsymbol{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L(\boldsymbol{p} \leftarrow \boldsymbol{\omega}_i) \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p)$$

$$I_i \in \{3\} \Rightarrow F = F + \frac{w_3(\boldsymbol{\omega}(\boldsymbol{x}_i))}{p(3) p_3(\boldsymbol{x}_i)} f_r(\boldsymbol{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L_e(\boldsymbol{x}_i \rightarrow -\boldsymbol{\omega}_i) G(\boldsymbol{p}, \boldsymbol{x}_i)$$

$F = \frac{1}{N} F$

A sample position from p_3 contributes to $\int_{\Omega_{\text{direct}}} \dots$. If not, then *V*=*G*=0.

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Combination of Ω_{direct} and $\Omega_{indirect}$

- MIS weights
  - E.g. $\boldsymbol{\omega} \in \Omega_{\text{indirect}} \Rightarrow w_1(\boldsymbol{\omega}) = w_2(\boldsymbol{\omega}) = 0.5$
    $\boldsymbol{\omega} \in \Omega_{\text{direct}} \Rightarrow w_1(\boldsymbol{\omega}) = w_2(\boldsymbol{\omega}) = 0$
    $\boldsymbol{\omega}(\boldsymbol{x}) \in \Omega_{\text{direct}} \Rightarrow w_3(\boldsymbol{\omega}(\boldsymbol{x})) = 1$
    $\boldsymbol{\omega}(\boldsymbol{x}) \in \Omega_{\text{indirect}} \Rightarrow w_3(\boldsymbol{\omega}(\boldsymbol{x})) = 0$

$F = 0$

Select $I_i \in \{1,2,3\}$ from p

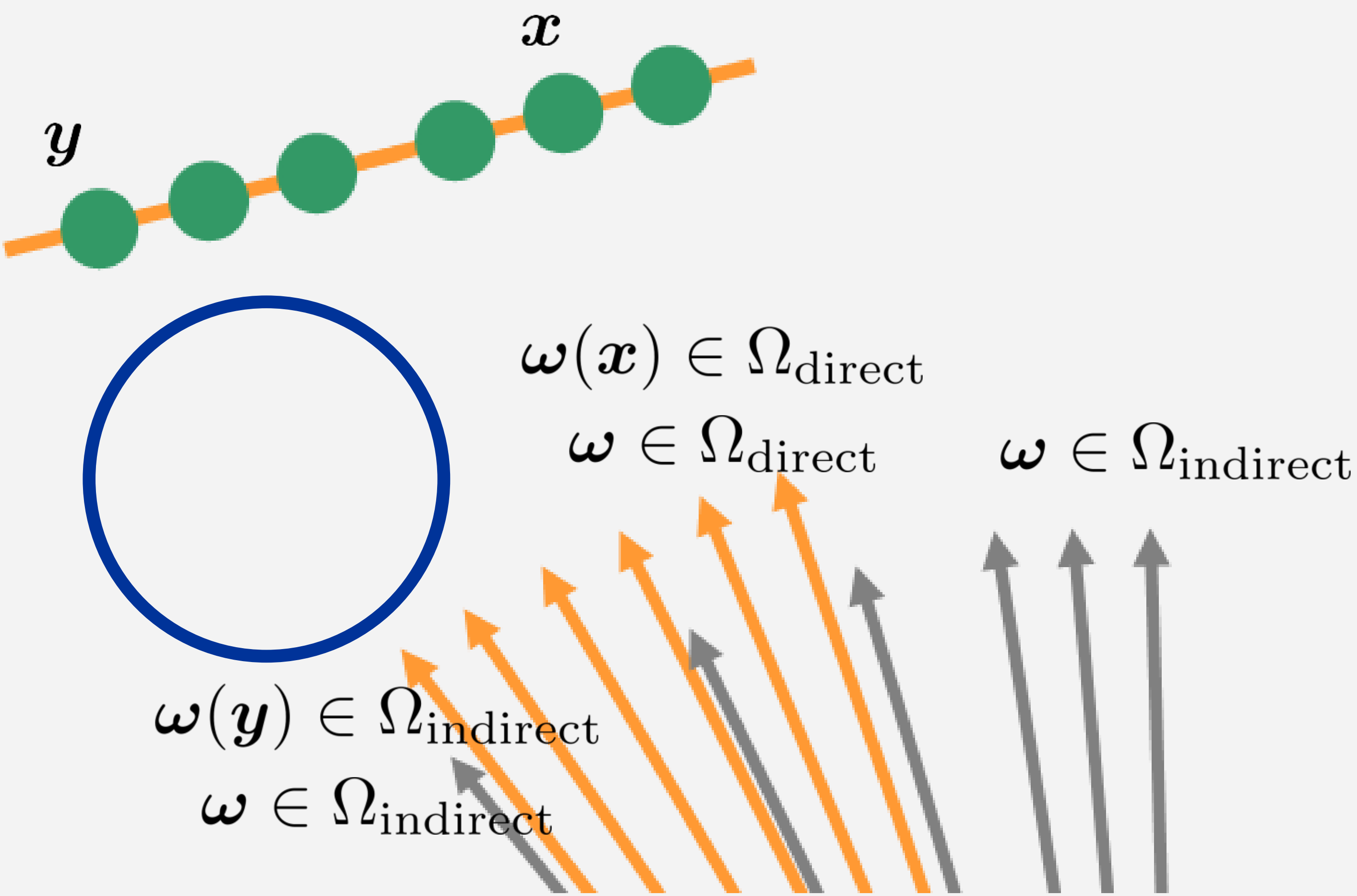
$I_i \in \{1,2\} \wedge \boldsymbol{\omega}_i \notin \Omega_{\text{direct}} \Rightarrow$

$$F = F + \frac{w_{I_i}(\boldsymbol{\omega}_i)}{p(I_i)p_{I_i}(\boldsymbol{\omega}_i)} f_r(\boldsymbol{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L(\boldsymbol{p} \leftarrow \boldsymbol{\omega}_i) \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p)$$

$$I_i \in \{3\} \Rightarrow F = F + \frac{1}{p(3)p_3(\boldsymbol{x}_i)} f_r(\boldsymbol{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L_e(\boldsymbol{x}_i \rightarrow -\boldsymbol{\omega}_i) G(\boldsymbol{p}, \boldsymbol{x}_i)$$

$F = \frac{1}{N} F$

Combination of Ω_{direct} and $\Omega_{indirect}$



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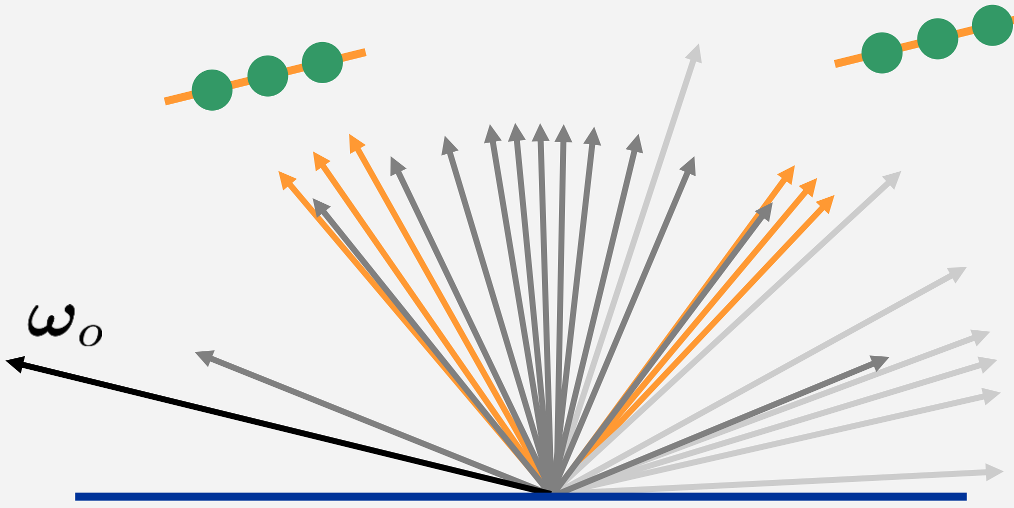
Combination of Ω_{direct} and $\Omega_{indirect}$

- If a ray in direction $\boldsymbol{\omega}$ does not hit a light source: $\boldsymbol{\omega} \in \Omega_{\text{indirect}}$
- If a ray in direction $\boldsymbol{\omega}$ hits a light source: $\boldsymbol{\omega} \in \Omega_{\text{direct}}$
- If a light source position $\boldsymbol{x}$ is visible from a surface point $\boldsymbol{p}$, the respective direction $\boldsymbol{\omega}(\boldsymbol{x})$ is in Ω_{direct}
- If a light source position $\boldsymbol{x}$ is not visible from a surface point $\boldsymbol{p}$, the respective direction $\boldsymbol{\omega}(\boldsymbol{x})$ is in Ω_{indirect}

Combination of Ω_{direct} and $\Omega_{indirect}$



Multiple importance sampling
for mixed material
and many small light sources

Combination of Ω_{direct} and $\Omega_{indirect}$

- Three PDFs
  - BRDF PDFs $p_1(\boldsymbol{\omega}_i) \propto \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p)$ $p_2(\boldsymbol{\omega}_i) \propto \cos(\boldsymbol{r}(\boldsymbol{n}_p, \boldsymbol{\omega}_i), \boldsymbol{\omega}_o)^e$
  - *k* light sources $p_3(I_k) = \frac{1}{k}$ $p_{I_k+3}(\boldsymbol{x}_i) = \frac{1}{A_{I_k}}$ $I_k \in \{1, \ldots, k\}$
  - Discrete PDF for PDF selection, e.g. $p(1) = p(2) = p(3) = \frac{1}{3}$

$F = 0$

Select $I_i \in \{1, 2, 3\}$ from p

$I_i \in \{1, 2\} \wedge \boldsymbol{\omega}_i \in \Omega_{\text{indirect}} \Rightarrow$

$F = F + \frac{w_{I_i}(\boldsymbol{\omega}_i)}{p(I_i) p_{I_i}(\boldsymbol{\omega}_i)} f_r(\boldsymbol{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L(\boldsymbol{p} \leftarrow \boldsymbol{\omega}_i) \cos(\boldsymbol{\omega}_i, \boldsymbol{n}_p)$

$I_i \in \{3\} \Rightarrow F = F + \frac{w_3(\boldsymbol{\omega}(\boldsymbol{x}_i))}{p(3) p_3(I_k) p_{I_k+3}(\boldsymbol{x}_i)} f_r(\boldsymbol{p}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) L_e(\boldsymbol{x}_i \rightarrow -\boldsymbol{\omega}_i) G(\boldsymbol{p}, \boldsymbol{x}_i)$

$F = \frac{1}{N} F$

Generate *N* samples

If a sample direction from p_1 or p_2 does not hit the light source, it contributes to $\int_{\Omega_{\text{indirect}}} \cdots$

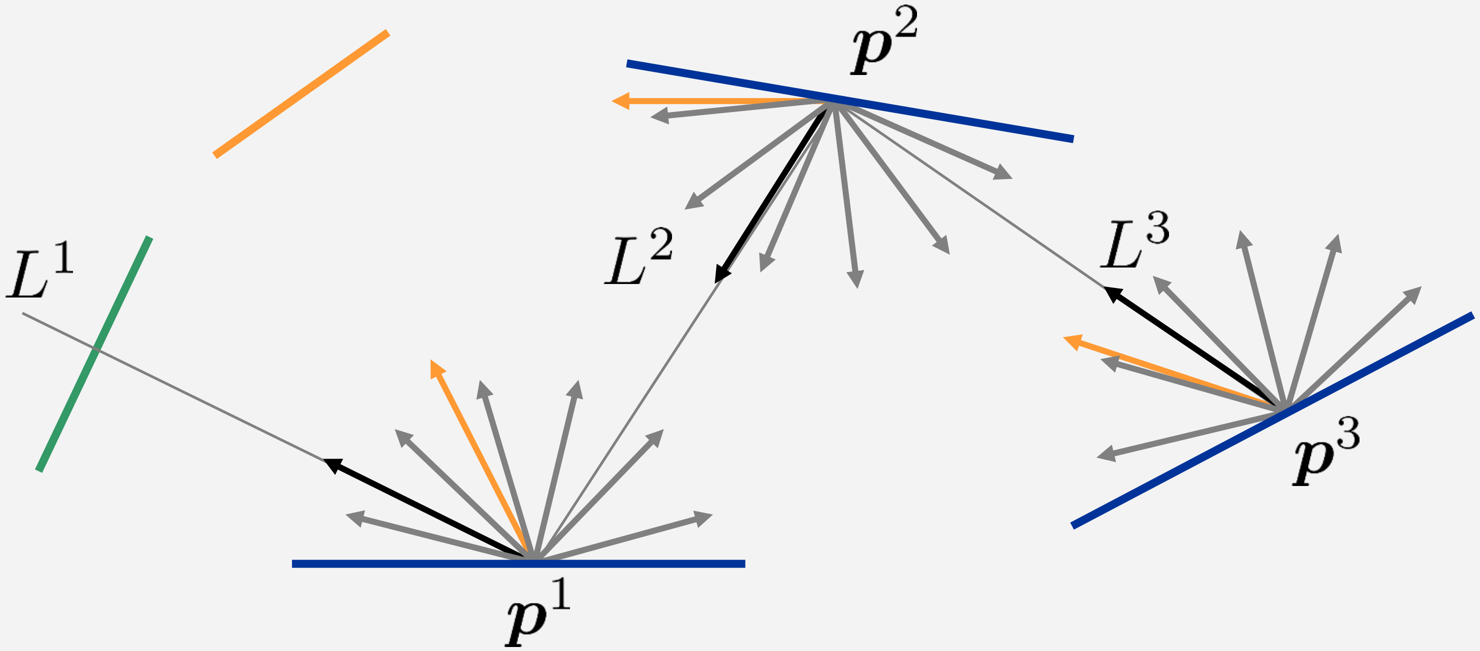
A sample position from p_3 contributes to $\int_{\Omega_{\text{direct}}} \cdots$

Computation of $\int_{\Omega_{\mathrm{indirect}}}$

– Notation: $L^1 = \int_{\Omega_{\mathrm{d}}^1} f_r^1 \cos^1 L_e^1 + \int_{\Omega_{\mathrm{i}}^1} f_r^1 \cos^1 L^2$

$$L^2 = \int_{\Omega_{\mathrm{d}}^2} f_r^2 \cos^2 L_e^2 + \int_{\Omega_{\mathrm{i}}^2} f_r^2 \cos^2 L^3$$

$$L^3 = \dots$$



Recursive Formulation

$$L^1 = \int_{\Omega_\mathrm{d}^1} f_r^1 \cos^1 L_e^1 + \int_{\Omega_\mathrm{i}^1} f_r^1 \cos^1 L^2$$

$$L^1 = \int_{\Omega_\mathrm{d}^1} f_r^1 \cos^1 L_e^1 + \int_{\Omega_\mathrm{i}^1} f_r^1 \cos^1 \left(\int_{\Omega_\mathrm{d}^2} f_r^2 \cos^2 L_e^2 + \int_{\Omega_\mathrm{i}^2} f_r^2 \cos^2 L^3 \right)$$

$$L^1 = \int_{\Omega_\mathrm{d}^1} f_r^1 \cos^1 L_e^1 + \int_{\Omega_\mathrm{i}^1} f_r^1 \cos^1 \left(\int_{\Omega_\mathrm{d}^2} f_r^2 \cos^2 L_e^2 + \int_{\Omega_\mathrm{i}^2} f_r^2 \cos^2 \left(\int_{\Omega_\mathrm{d}^3} f_r^3 \cos^3 L_e^3 + \int_{\Omega_\mathrm{i}^3} f_r^3 \cos^3 L^4 \right)\right)$$

– Recursion is terminated by setting $\int_{\Omega_\mathrm{i}^k} = 0$, e.g.

$$L^1 = \int_{\Omega_\mathrm{d}^1} f_r^1 \cos^1 L_e^1 + \int_{\Omega_\mathrm{i}^1} f_r^1 \cos^1 \left(\int_{\Omega_\mathrm{d}^2} f_r^2 \cos^2 L_e^2 + \int_{\Omega_\mathrm{i}^2} f_r^2 \cos^2 \left(\int_{\Omega_\mathrm{d}^3} f_r^3 \cos^3 L_e^3 \right)\right)$$

$$L^1 = \int_{\Omega_\mathrm{d}^1} f_r^1 \cos^1 L_e^1 + \int_{\Omega_\mathrm{i}^1} f_r^1 \cos^1 \int_{\Omega_\mathrm{d}^2} f_r^2 \cos^2 L_e^2 + \int_{\Omega_\mathrm{i}^1} f_r^1 \cos^1 \int_{\Omega_\mathrm{i}^2} f_r^2 \cos^2 \int_{\Omega_\mathrm{d}^3} f_r^3 \cos^3 L_e^3 + \ldots$$

Emitted light after one bounce

Emitted light after two bounces

Emitted light after three bounces

Emitted light after more bounces

Path Tracing

$$L^1 = \int_{\Omega_\mathrm{d}^1} f_r^1 \cos^1 L_e^1 + \int_{\Omega_\mathrm{i}^1} f_r^1 \cos^1 \int_{\Omega_\mathrm{d}^2} f_r^2 \cos^2 L_e^2 + \int_{\Omega_\mathrm{i}^1} f_r^1 \cos^1 \int_{\Omega_\mathrm{i}^2} f_r^2 \cos^2 \int_{\Omega_\mathrm{d}^3} f_r^3 \cos^3 L_e^3 + \ldots$$

$$L^1 \approx \frac{1}{N_{d,1}} \sum_{i=1}^{N_{d,1}} \frac{f_r^1 \cos^1 L_e^1}{\mathrm{pdf}_d^1} + \frac{1}{N_{i,1}} \sum_{i=1}^{N_{i,1}} \frac{f_r^1 \cos^1}{\mathrm{pdf}_i^1} \left(\frac{1}{N_{d,2}} \sum_{i=1}^{N_{d,2}} \frac{f_r^2 \cos^2}{\mathrm{pdf}_d^2} L_e^2 \right) + \ldots$$

– Taking one sample everywhere

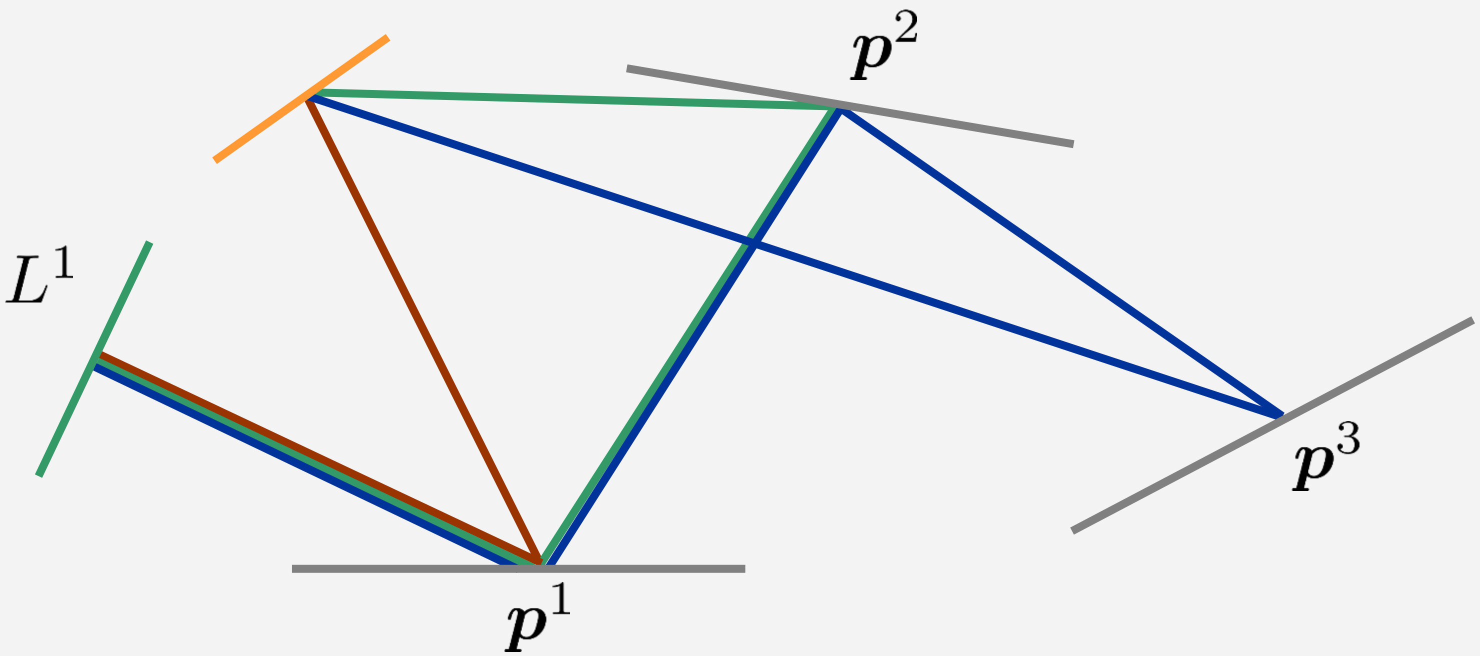
$$L^1 \approx \frac{f_r^1 \cos^1}{\mathrm{pdf}_d^1} L_e^1 + \frac{f_r^1 \cos^1}{\mathrm{pdf}_i^1} \frac{f_r^2 \cos^2}{\mathrm{pdf}_d^2} L_e^2 + \frac{f_r^1 \cos^1}{\mathrm{pdf}_i^1} \frac{f_r^2 \cos^2}{\mathrm{pdf}_i^2} \frac{f_r^3 \cos^3}{\mathrm{pdf}_d^3} L_e^3 + \ldots$$

Path tracing with next event estimation

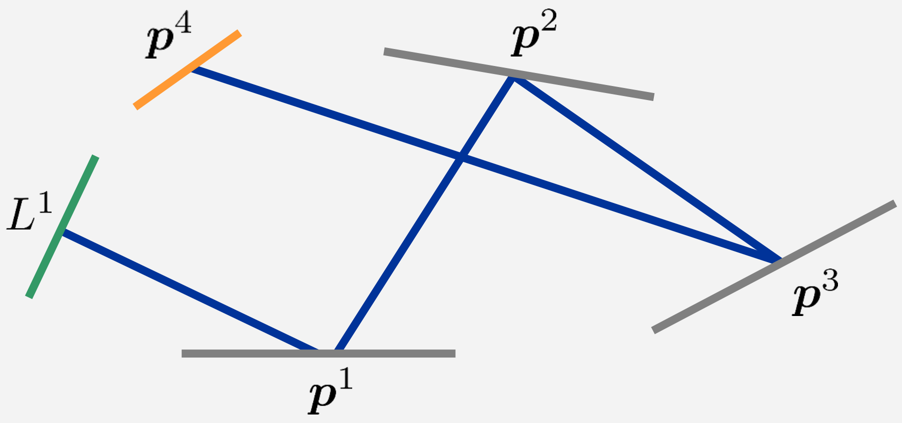
Path Tracing with Next Event Estimation

$$L^1 \approx \frac{f_r^1 \cos^1}{\mathrm{pdf}_d^1} L_e^1 + \frac{f_r^1 \cos^1}{\mathrm{pdf}_i^1} \frac{f_r^2 \cos^2}{\mathrm{pdf}_d^2} L_e^2 + \frac{f_r^1 \cos^1}{\mathrm{pdf}_i^1} \frac{f_r^2 \cos^2}{\mathrm{pdf}_i^2} \frac{f_r^3 \cos^3}{\mathrm{pdf}_d^3} L_e^3 + \dots$$

Path 1 Path 2 Path 3



General Path Tracing



$$L^1 = 0$$

Generate a sample towards p_1

$$\beta = 1 \qquad L_e > 0 \Rightarrow L^1 = L^1 + \beta L_e$$

Generate a sample from p_1 into the entire hemisphere with pdf_1 towards p_2

$$\beta = \beta \cdot \frac{f_r^1 \cos^1}{\text{pdf}^1} \qquad L_e > 0 \Rightarrow L^1 = L^1 + \beta L_e$$

Generate a sample from p_2 into the entire hemisphere with pdf_2 towards p_3

$$\beta = \beta \cdot \frac{f_r^2 \cos^2}{\text{pdf}^2} \qquad L_e > 0 \Rightarrow L^1 = L^1 + \beta L_e$$

Generate a sample towards p_4

$$\beta = \beta \cdot \frac{f_r^3 \cos^3}{\text{pdf}^3} \qquad L_e > 0 \Rightarrow L^1 = L^1 + \beta L_e$$

Terminate, if e.g. throughput β smaller than user-defined threshold

Assumes reflective light sources.

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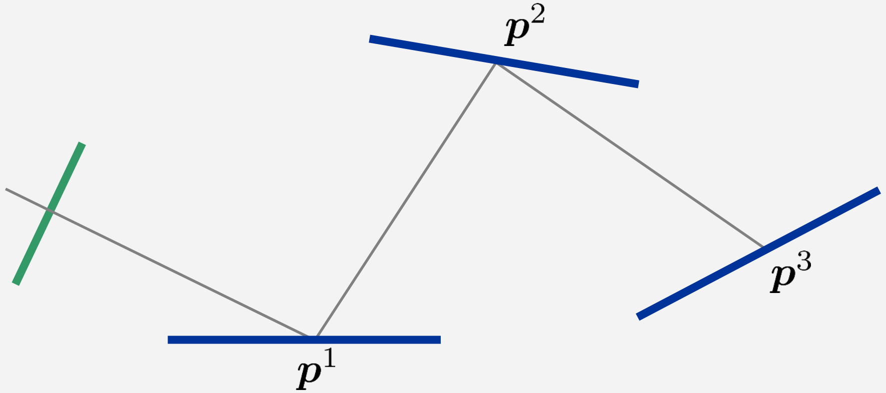
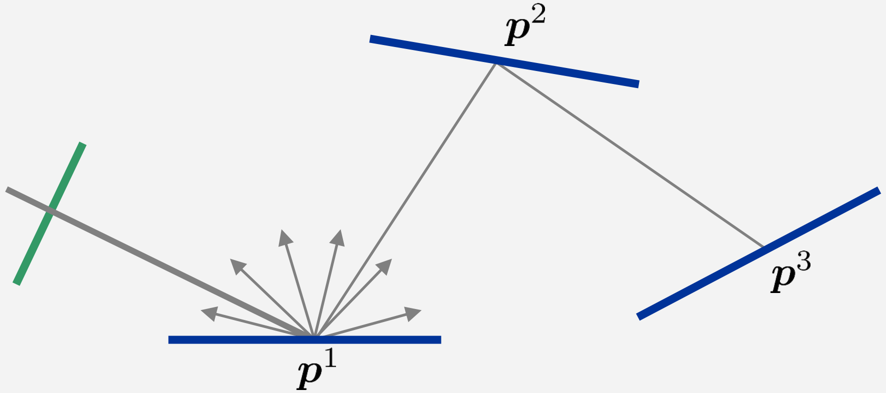
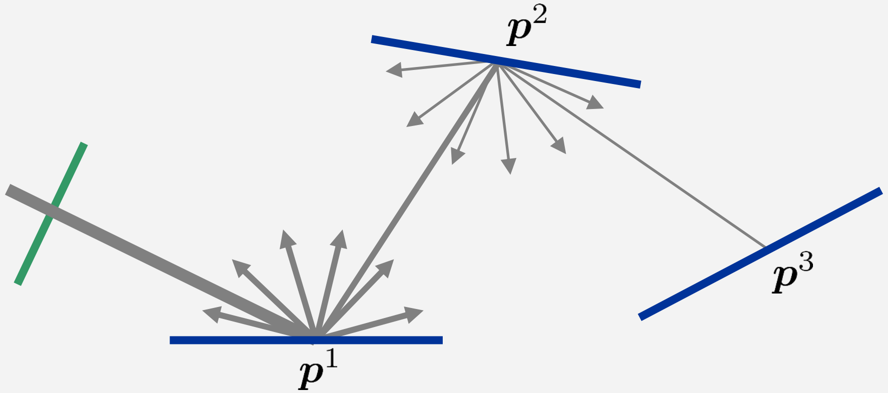
General Path Tracing



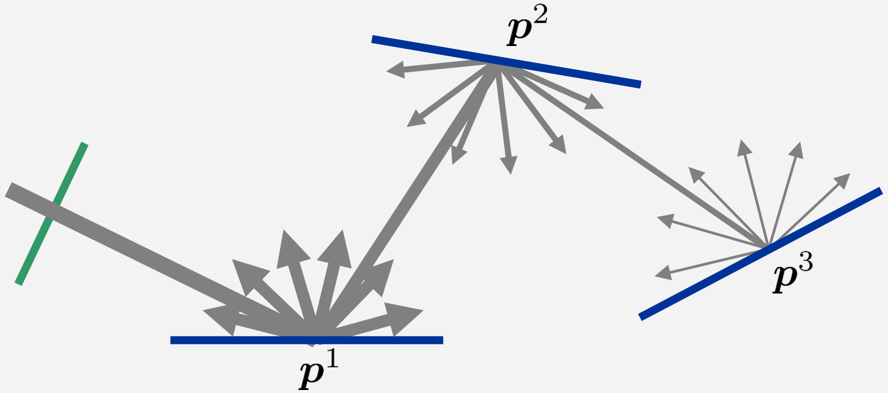
1 path per pixel.



n paths per pixel. Sample distribution at first bounce converges to *pdf*^{1}.



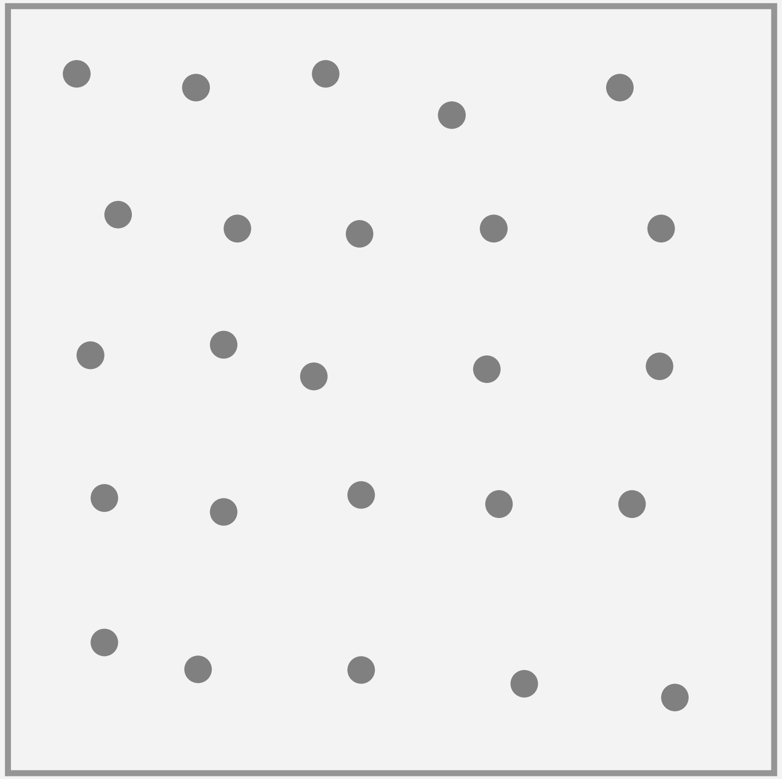
n·n paths per pixel. Sample distribution at second bounce converges to *pdf*^{2}.



n·n·n paths per pixel. Sample distribution at third bounce converges to *pdf*^{3}.

Jittered Path Sampling per Pixel

- Various strategies
  - Random
  - Stratified
  - Quasi random (low-discrepancy sequences)
- Regular sampling would cause aliasing



n samples / paths per pixel.

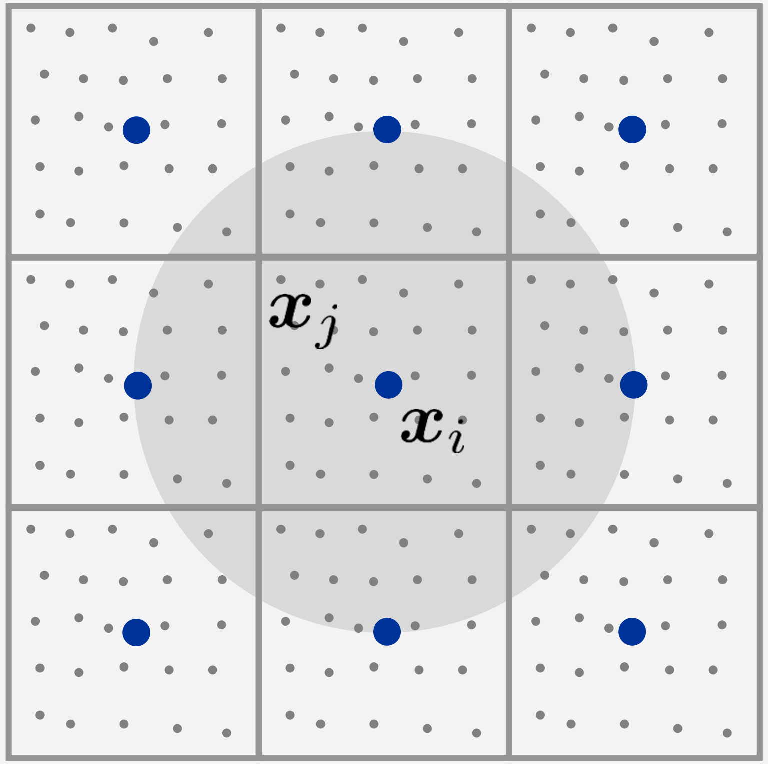
Image Reconstruction

- Convolution with a normalized kernel function / filter *W*

$$L(\boldsymbol{x}_i) = \int_A L(\boldsymbol{x}')W(\|\boldsymbol{x}_i - \boldsymbol{x}'\|)\mathrm{d}\boldsymbol{x}'$$

$$L(\boldsymbol{x}_i) \approx \sum_j L(\boldsymbol{x}_j)W(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|)A(\boldsymbol{x}_j)$$

- E.g., box filter: $W = \text{const}$
  - $\boldsymbol{x}_j$ - uniform samples in one pixel *i*
  - $A = \frac{\text{area of pixel}}{N}$ $\quad \sum_j W = \frac{1}{\text{area of pixel}}$
  - $\sum_j W \cdot A = \frac{\text{area of pixel}}{N} \cdot \frac{1}{\text{area of pixel}}$
  - $L(\boldsymbol{x}_i) = \sum_j L(\boldsymbol{x}_j) \cdot W \cdot A = \frac{1}{N} \sum_j L(\boldsymbol{x}_j)$



9 pixels with *N* samples. Pixel *i* with representative position x_i and samples / paths at positions x_j

Path Tracing – Maximum Path Length

- Fixed, user-defined
- Adaptive with Russian Roulette
  - Minimum fixed length, user-defined
  - Termination probability q for additional segments
  - Random sample ξ :

$$F' = \begin{cases} \frac{F-qc}{1-q} & \xi > q \\ c & \text{otherwise} \end{cases}$$

Typically c=0

  - Intuition
    - Some samples are discarded
    - Remaining samples are amplified to account for missing contributions

$$E[F'] = (1-q)\frac{E[F]-qc}{1-q} + qc$$
$$= E[F]$$

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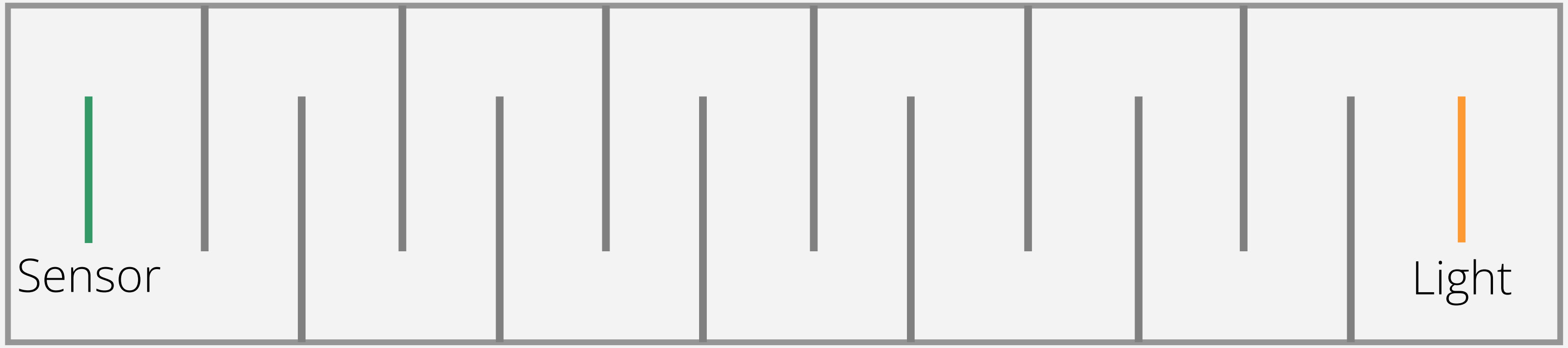
Russian Roulette - Motivation

Fixed path length

- Biased estimator
- Always too small / dark, but consistent
- Converges to correct result
- Completely misses effects that require longer paths
- Example: All paths with zero contribution

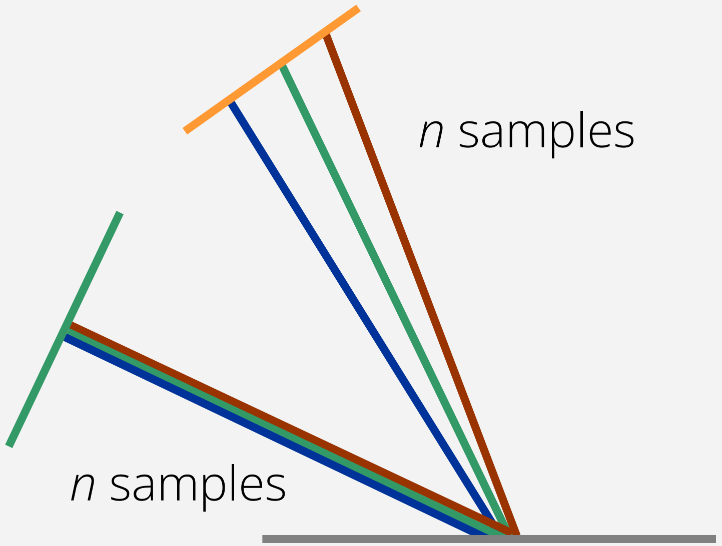
Adaptive path length with Russian Roulette

- Unbiased estimator
- Converges to correct result
- Arbitrarily long paths potentially capture more effects than fixed path lengths, although with low quality
- Example: Most samples with zero contribution Some very long paths with non-zero contr.

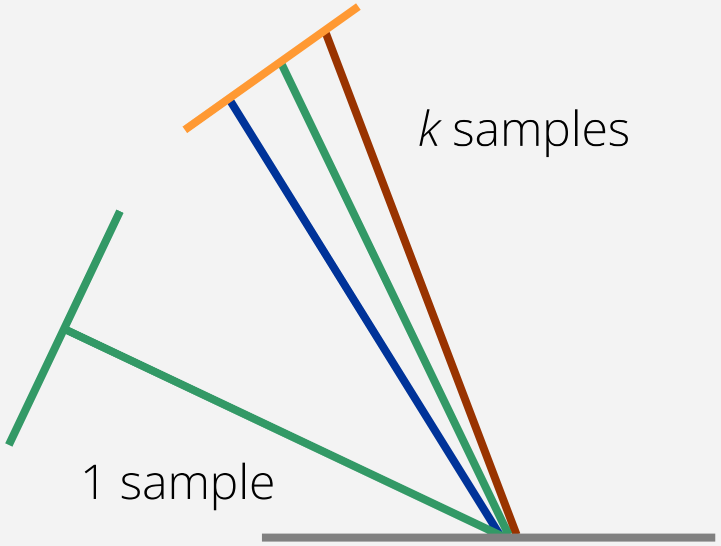


Splitting

– Adaptive sample counts at intersections



Without splitting



Splitting

Path Tracing – Degrees of Freedom

- $L = \prod_{i=1}^{N} \frac{f_r^i \cos^i}{\mathrm{pdf}^i} L_e^N$
- Path length, e.g. Russian Roulette
- PDFs at each intersection, e.g. MIS with light and material sampling
- Next event estimation, i.e. light sampling at each intersection
- Number of samples at each intersection, i.e. splitting

Current Variants

- Bidirectional path generation
  - Motivation: Samples from the light source into the scene are as important as samples from the sensor into the scene
  - Symmetric setting
- Metropolis sampling
  - Path mutations instead of random sampling
  - Small mutations in case of relevant paths
  - Large mutations in case of less relevant paths