Advanced Computer Graphics
Sampling Strategies for Solving the Rendering Equation

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Outline

- Context
- Some concepts
- Direct illumination
- Indirect illumination
Goal and Governing Equation

- Computation of incident radiance at a sensor $L(s \leftarrow \omega_s)$
- Incident radiance at sensor position $s$ is equal to exitant radiance at scene position $p$ with $p = r_c(s, \omega_s)$:
  
  $$L(s \leftarrow \omega_s) = L(p \rightarrow \omega_o)$$
  
- Raycast operator $r_c$, conservation of radiance

- Exitant radiance at scene position $p$ is computed as:
  
  $$L(p \rightarrow \omega_o) = L_e(p \rightarrow \omega_o) + \int_{\Omega} f_r(p, \omega_i \leftrightarrow \omega_o)L(p \leftarrow \omega_i)\cos(\omega_i, n_p)\,d\omega_i$$
  
- Rendering equation
Goal and Governing Equation

\[ L(s \leftarrow \omega_s) = L(p \rightarrow \omega_o) = L_e(p \rightarrow \omega_o) + \int_{\Omega} f_r(p, \omega_i \leftrightarrow \omega_o) L(p \leftarrow \omega_i) \cos(\omega_i, n_p) d\omega_i \]

[Diagram showing light paths and equations related to light scattering and reflection.]
Monte Carlo Integration

\[ L(p \rightarrow \omega_o) = L_e(p \rightarrow \omega_o) + \int_{\Omega} f_r(p, \omega_i \leftrightarrow \omega_o)L(p \leftrightarrow \omega_i)\cos(\omega_i, n_p)\,d\omega_i \]

is approximated with

\[ L(p \rightarrow \omega_o) = L_e(p \rightarrow \omega_o) + \sum_{i=1}^{N} f_r(p, \omega_i \leftrightarrow \omega_o)L(p \leftrightarrow \omega_i)\cos(\omega_i, n_p)\frac{1}{N_p(\omega_i)} \]

– \(N\) randomly sampled directions \(\omega_i\)
– According to a probability density function \(p(\omega_i)\)
Monte Carlo Integration

\[\int_{\Omega} f_r(p, \omega_i \leftrightarrow \omega_o) L(p \leftarrow \omega_i) \cos(\omega_i, n_p) d\omega_i\]

\[\approx \sum_{i=1}^{N} f_r(p, \omega_i \leftrightarrow \omega_o) L(p \leftarrow \omega_i) \cos(\omega_i, n_p) \frac{1}{Np(\omega_i)}\]
Monte Carlo Integration – Error

- Estimated sample size is not equal to the actual sample size due to random sample selection
- Sample contributions are randomly over- or underestimated

Uniform sampling of a 3D hemisphere

Estimated sample size

Uniform random sampling of a 3D hemisphere

Actual sample size
Monte Carlo Integration - Error

- **Variance**, noise: resulting radiance values are randomly too dark or too bright
- If a Monte Carlo approximation converges for growing sample numbers to the correct result, the scheme is **unbiased**, otherwise **biased**
Monte Carlo Integration - Variance

8 samples per pixel

1024 samples per pixel

[Pharr et al., Physically Based Rendering]
Rendering Equation – Recursive Problem

– Solving the rendering equation requires $N$ samples, where many samples require the solution of another rendering equation.

$O(N)$ samples

$O(N \cdot N)$ samples

$O(N \cdot N \cdot N)$ samples
Need of a Sampling Strategy

- Sample processing is expensive
  - Ray-scene intersection tests
- Samples differ in terms of relevance
- Important samples, e.g.
  - Towards / from visible light sources
  - From / towards sensors
  - Towards / from bright parts of a scene
- Less important samples, e.g.
  - After increasing number of bounces
  - Towards / from dark parts in a scene
Outline

- Context
- Some concepts
- Direct illumination
- Indirect illumination
Stratification

– Subdivision of the integration domain, e.g.

\[ L(p \rightarrow \omega_o) = L_e(p \rightarrow \omega_o) + \int_{\Omega} f_r(p, \omega_i \leftrightarrow \omega_o)L(p \leftarrow \omega_i) \cos(\omega_i, n_p)\,d\omega_i \]

\[ = L_e(p \rightarrow \omega_o) + \int_{\Omega_{\text{direct}}} f_r(p, \omega_i \leftrightarrow \omega_o)L_e(p \leftarrow \omega_i) \cos(\omega_i, n_p)\,d\omega_i \]

\[ + \int_{\Omega_{\text{indirect}}} f_r(p, \omega_i \leftrightarrow \omega_o)L(p \leftarrow \omega_i) \cos(\omega_i, n_p)\,d\omega_i \]

– Integral over \( \Omega_{\text{direct}} \) can directly be computed using \( L_e \)

– Integral over \( \Omega_{\text{indirect}} \) requires the recursive computation of \( L \)
Stratification

Two sample sets for two parts of the integration domain

Light source

$\Omega_{\text{direct}}$

$\Omega_{\text{indirect}}$

Light source

$\Omega_{\text{direct}}$

$\Omega_{\text{indirect}}$

$\Omega_{\text{indirect}}$
Stratification

- Subdivision into non-overlapping strata
  - Allows the usage of an individual technique for each stratum
  - Allows / requires the individual sampling of each stratum
  - Avoids sample clustering in a part of the integration domain
Importance Sampling

- Probability density function
  - Should be proportional to the integrand
    \[ p(\omega_i) \propto f_r(p, \omega_i \leftrightarrow \omega_o)L(p \leftarrow \omega_i)\cos(\omega_i, n_p) \]
  - Product of functions
  - Incident radiance expensive to compute
- Optimal PDF
  \[ p(\omega_i) = \frac{f_r(p, \omega_i \leftrightarrow \omega_o)L(p \leftarrow \omega_i)\cos(\omega_i, n_p)}{\int_{\Omega} f_r(p, \omega_i \leftrightarrow \omega_o)L(p \leftarrow \omega_i)\cos(\omega_i, n_p)d\omega_i} \]
  - Irrelevant: If the integral would be known, we are done.
Importance Sampling

– Large integrand values
  – More samples with smaller size and reduced sampling inaccuracies to improve accuracy, i.e. minimize variance / noise

– Small integrand values
  – Less samples with larger size and larger sampling errors to improve efficiency
Multiple Importance Sampling MIS

- Combine sample sets from different PDFs

\[
\int_\Omega f(x) \, dx \approx \sum_{j=1}^{M} f(X_j) \frac{1}{M p(X_j)}
\]

Monte Carlo with \(M\) samples

\[
= \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{M} f(X_j) \frac{1}{M p(X_j)}
\]

Summing up \(N\) MC estimates and dividing by \(N\)

\[
= \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N_i} f(X_i,j) \frac{1}{N_i p_i(X_i,j)}
\]

Using individual PDFs \(p_i\) with individual sample counts \(N_i\) for each of the \(N\) MC estimates

\[
= \sum_{j=1}^{N_1} \frac{1}{N} f(X_{1,j}) \frac{1}{N_1 p_1(X_{1,j})} + \sum_{j=1}^{N_2} \frac{1}{N} f(X_{2,j}) \frac{1}{N_2 p_2(X_{2,j})} + \ldots + \sum_{j=1}^{N_N} \frac{1}{N} f(X_{N,j}) \frac{1}{N_N p_N(X_{N,j})}
\]

Replacing weight \(1/N\) with individual weighting functions \(w_i\)

\[
= \sum_{j=1}^{N_1} w_1(X_{1,j}) f(X_{1,j}) \frac{1}{N_1 p_1(X_{1,j})} + \sum_{j=1}^{N_2} w_2(X_{2,j}) f(X_{2,j}) \frac{1}{N_2 p_2(X_{2,j})} + \ldots
\]

\[
= \sum_{i=1}^{N} \sum_{j=1}^{N_i} w_i(X_i,j) f(X_i,j) \frac{1}{N_i p_i(X_i,j)} = \sum_{i=1}^{N} \frac{1}{N_i} \sum_{j=1}^{N_i} w_i(X_i,j) \frac{f(X_i,j)}{p_i(X_i,j)}
\]
Multiple Importance Sampling MIS

\[ \int_{\Omega} f(x)dx \approx \sum_{i=1}^{N} \frac{1}{N_i} \sum_{j=1}^{N_i} w_i(X_{i,j}) \frac{f(X_{i,j})}{p_i(X_{i,j})} \]

- Use \( N \) PDFs \( p_i \)
- Generate \( N_i \) samples \( X_{i,j} \) from PDF \( p_i \)
- Weight all contributions with functions \( w_i(x) : \Omega \to \mathbb{R} \)
  - Constraints for weighting functions
    
    \[ f(x) \neq 0 \Rightarrow \sum_i w_i(x) = 1 \]
    
    \[ p_j(x) = 0 \Rightarrow w_j(x) = 0 \]
    
    \[ \Rightarrow \sum_{i \neq j} w_i(x) = 1 \]

The weights have to add up to one everywhere on \( \Omega \). The weights are irrelevant, if a sample has zero contribution. If any of the PDFs is zero for some \( x \), the weights for all other PDFs have to sum up to one.
MIS – Example Weightings

\[ \int_{\Omega} f(x) dx \approx \sum_{i=1}^{N} \frac{1}{N_i} \sum_{j=1}^{N_i} w_i(X_{i,j}) \frac{f(X_{i,j})}{p_i(X_{i,j})} \]

\( w_i(x) \in \{0, 1\} \quad \sum_i w_i(x) = 1 \)

\( X_{i,j} \in \Omega_{\text{indirect}} \Rightarrow w_1(X_{i,j}) = 1 \land w_2(X_{i,j}) = 0 \)

\( X_{i,j} \in \Omega_{\text{direct}} \Rightarrow w_1(X_{i,j}) = 0 \land w_2(X_{i,j}) = 1 \)

- Realizes stratification
- E.g. generate samples from \( p_1 \) and \( p_2 \)
- Use a sample from \( p_1 \), if it is in \( \Omega_{\text{indirect}} \) and discard it if it is in \( \Omega_{\text{direct}} \)
- Use a sample from \( p_2 \), if it is in \( \Omega_{\text{direct}} \) and discard it if it is in \( \Omega_{\text{indirect}} \)
MIS – Example Weightings

\[ \int_{\Omega} f(x) \, dx \approx \sum_{i=1}^{N} \frac{1}{N_i} \sum_{j=1}^{N_i} w_i(X_{i,j}) \frac{f(X_{i,j})}{p_i(X_{i,j})} \]

\[ w_i(X_{i,j}) = \frac{1}{N} \]

– Compute \( N \) MC estimates with PDFs \( p_i \) and average them

\[ w_i(X_{i,j}) = \frac{p_i(X_{i,j})}{\sum_{k=1}^{N} p_k(X_{i,j})} \]


– Larger weight to more accurate samples with smaller size

– Good, if any of the \( p_i \) is large for large \( f \), but no \( p_i \) is proportional to \( f \) everywhere

– If any \( p_i \) is perfectly proportional to \( f \), the balance heuristic is not optimal
MIS – Example Weightings

\[ \int_{\Omega} f(x) \, dx \approx \sum_{i=1}^{N} \frac{1}{N_i} \sum_{j=1}^{N_i} w_i(X_{i,j}) \frac{f(X_{i,j})}{p_i(X_{i,j})} \]

\[ w_i(X_{i,j}) = \frac{\sum_{k=1}^{N} p_i(X_{i,j})^\beta}{\sum_{k=1}^{N} p_k(X_{i,j})^\beta} \quad \beta = 2 \]

- Power heuristic [Eric Veach 1995, 1997]
- Popular choice in MIS

- Other alternatives
  - Cutoff heuristic
  - Maximum heuristic
MIS – Adaptive Sample Counts

\[ \int_{\Omega} f(x)dx \approx \sum_{i=1}^{N} \frac{1}{N_i} \sum_{j=1}^{N_i} w_i(X_{i,j}) \frac{f(X_{i,j})}{p_i(X_{i,j})} \]

- Fixed sample counts \( N_i \) can be replaced by randomly selecting a PDF \( p_i \) from a discrete PDF \( p(i) \)
- One-sample estimator
  - Generate \( I_k \) from \( p \)
  - Generate \( X_k \) from \( p_{I_k} \)

\[ \int_{\Omega} f(x)dx \approx \frac{1}{N} \sum_{k=1}^{N} \frac{w_{I_k}(X_k)f(X_k)}{p(I_k)p_{I_k}(X_k)} \approx \frac{w_{I_1}(X_{I_1})f(X_1)}{p(I_1)p_{I_1}(X_1)} \]

- Relevant, e.g. in path tracing
MIS - Example

- Diffuse material under direct illumination $L_e$
  - Regular importance sampling with a PDF $p_1(\omega_i) \propto \cos(\omega_i, n_p)$
    \[
    \int_\Omega \frac{\rho_d}{\pi} L_e \cos(\omega_i, n_p) d\omega_i \approx \sum_{i=1}^N \frac{\rho_d}{\pi} L_e \cos(\omega_i, n_p) \frac{1}{N p_1(\omega_i)}
    \]

- Mixed material under $L_e$
  - Multiple importance sampling with two PDFs $p_1$ and $p_2$
    with $p_1(\omega_i) \propto \cos(\omega_i, n_p)$ and $p_2(\omega_i) \propto \cos(r(n_p, \omega_i), \omega_o)^e$
    \[
    \int_\Omega \left( \frac{\rho_d}{\pi} + \rho_g \cos(r(n_p, \omega_i), \omega_o)^e \right) L_e \cos(\omega_i, n_p) d\omega_i \quad r - \text{reflection direction}
    \approx \frac{1}{N} \sum_{i=1}^N \frac{w_{I_i}(\omega_i)}{p(I_i)p_{I_i}(\omega_i)} \left( \frac{\rho_d}{\pi} + \rho_g \cos(r(n_p, \omega_i), \omega_o)^e \right) L_e \cos(\omega_i, n_p)
    \]
    $I_i \in \{1, 2\}$ from $p$, e.g. $p(1) = p(2) = \frac{1}{2}$, $\omega_i$ from $p_{I_i}$
MIS - Example

Importance sampling for a diffuse surface
Using samples from one PDF

Multiple importance sampling for mixed material
Using two sample sets from two PDFs
Weighted averaging of two MC estimates
Outline

– Context
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Problem

- Computation of $\int_{\Omega_{\text{direct}}}$ from the rendering equation

$$L_o(p \rightarrow \omega_o) = L_e(p \rightarrow \omega_o) + \int_{\Omega_{\text{direct}}} f_r(p, \omega_i \leftrightarrow \omega_o) L_e(p \leftarrow \omega_i) \cos(\omega_i, n_p) d\omega_i$$

Emitted light

$$+ \int_{\Omega_{\text{indirect}}} f_r(p, \omega_i \leftrightarrow \omega_o) L(p \leftarrow \omega_i) \cos(\omega_i, n_p) d\omega_i$$

Emitted light after 1 bounce

Emitted light after $n$ bounces with $n > 1$
Hemisphere Dominated by $L_e$

- BRDF sampling
- Sampling directions from a PDF proportional to the BRDF
- Diffuse: $p_1(\omega_i) \propto \cos(\omega_i, n_p)$
  \[ \sum_{i=1}^{N} \frac{\rho_d}{\pi} L_e \cos(\omega_i, n_p) \frac{1}{N p_1(\omega_i)} \]
- Mixed: $p_1(\omega_i) \propto \cos(\omega_i, n_p) \quad p_2(\omega_i) \propto \cos(r(n_p, \omega_i), \omega_o)^e$
  \[ \frac{1}{N} \sum_{i=1}^{N} \frac{w_{I_i}(\omega_i)}{p(I_i)p_{I_i}(\omega_i)} \left( \frac{\rho_d}{\pi} + \rho_g \cos(r(n_p, \omega_i), \omega_o)^e \right) L_e \cos(\omega_i, n_p) \]
- Majority of samples hit a light source, only few misses with zero contribution
Small Light Source

- Majority of samples would miss in case of BRDF sampling, inefficient
- Light sampling
  - Use area form of the rendering equation
  - Sample positions on the light source instead of directions

\[
\int_{\Omega_{\text{direct}}} f_r(p, \omega_i \leftrightarrow \omega_o) L_e(p \leftrightarrow \omega_i) \cos(\omega_i, n_p) d\omega_i \\
= \int_{A_{\text{direct}}} f_r(p, \omega_i \leftrightarrow \omega_o) L_e(x \rightarrow -\omega_i) V(p, x) \frac{\cos(\omega_i, n_p) \cos(-\omega_i, n_x)}{r_{px}^2} dx
\]
Light Sampling

\[ \int_{A_{\text{direct}}} f_r(p, \omega_i \leftrightarrow \omega_o) L_e(x \rightarrow -\omega_i) V(p, x) \frac{\cos(\omega_i, n_p) \cos(-\omega_i, n_x)}{r_{px}^2} \, dx \]

\[ = \int_{A_{\text{direct}}} f_r(p, \omega_i \leftrightarrow \omega_o) L_e(x \rightarrow -\omega_i) G(p, x) \, dx \]

\[ \approx \sum_{i=1}^{N} f_r(p, \omega_i \leftrightarrow \omega_o) L_e(x_i \rightarrow -\omega_i) G(p, x_i) \frac{1}{N_p(x_i)} \]

– E.g., uniform light sampling \( p(x_i) = \frac{1}{A_{\text{direct}}} \)

– Area of the light source \( A_{\text{direct}} \)

– Position \( x_i \) is sampled, direction \( \omega_i \)

is computed as \( \omega_i = \frac{x_i - p}{\|x_i - p\|} \)
Many Small Light Sources

- $N_l$ light sources with areas $A_{\text{direct},j}$
- Uniform sampling of all light sources

$$\sum_{j=1}^{N_l} \sum_{i=1}^{N} f_r(p, \omega_{j,i} \leftrightarrow \omega_o) L_e(x_{j,i} \rightarrow -\omega_{j,i}) G(p, x_{j,i}) \frac{1}{Np_j(x_{j,i})}$$
Adaptive Sample Counts

- Random light source selection from a discrete PDF $p$
- One-sample estimator
  - Generate $I_k$ from $p$
  - Generate positions $x_k$ from $p_{I_k}$
  - Compute $\omega_k = \frac{x_k - p}{\|x_k - p\|}$
  - MC estimator
    $$\sum_{k=1}^{N} f_r(p, \omega_k \leftrightarrow \omega_o) L_e(x_k \rightarrow -\omega_k) G(p, x_k) \frac{1}{Np(I_k)p_{I_k}(x_k)}$$
    $$\approx f_r(p, \omega_1 \leftrightarrow \omega_o) L_e(x_1 \rightarrow -\omega_1) G(p, x_1) \frac{1}{p(I_1)p_{I_1}(x_1)}$$
- Relevant, e.g. in path tracing
Light Source Sampling

- Random light source selection
  - Based on relevance for $\int_{\Omega_{\text{direct}}} \ldots$
  - Discrete PDF $p$ should be proportional to
    - Projected light source area
    - Light source power
- Sampling of each light source
  - Proportional to spatial power distribution
Outline

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Problem

– Computation of $\int_{\Omega_{\text{indirect}}} f$ from the rendering equation

$$L_o(p \rightarrow \omega_o) = L_e(p \rightarrow \omega_o) + \int_{\Omega_{\text{direct}}} f_r(p, \omega_i \leftrightarrow \omega_o) L_e(p \leftarrow \omega_i) \cos(\omega_i, n_p) d\omega_i$$

Emitted light  Emitted light after 1 bounce

$$+ \int_{\Omega_{\text{indirect}}} f_r(p, \omega_i \leftrightarrow \omega_o) L(p \leftarrow \omega_i) \cos(\omega_i, n_p) d\omega_i$$

Emitted light after $n$ bounces with $n>1$
Combination of $\Omega_{\text{direct}}$ and $\Omega_{\text{indirect}}$

- $\int \Omega_{\text{direct}}$ can already be computed
- Assume, $\int \Omega_{\text{indirect}}$ can also be computed

Multiple importance sampling for mixed material and a small light source

Using three sample sets from three PDFs
Weighted averaging of three MC estimates Relevant for recursive raytracing to compute
$$\int \Omega_{\text{direct}} \ldots + \int \Omega_{\text{indirect}} \ldots$$
Combination of $\Omega_{\text{direct}}$ and $\Omega_{\text{indirect}}$

- Three PDFs
  - BRDF PDFs $p_1(\omega_i) \propto \cos(\omega_i, n_p)$ $p_2(\omega_i) \propto \cos(r(n_p, \omega_i), \omega_o)^\epsilon$
  - Light PDF $p_3(x_i) = \frac{1}{A_{\text{direct}}}$
  - Discrete PDF for PDF selection, e.g. $p(1) = p(2) = p(3) = \frac{1}{3}$
    
    $F = 0$
    
    Select $I_i \in \{1, 2, 3\}$ from $p$
    
    $I_i \in \{1, 2\} \land \omega_i \in \Omega_{\text{indirect}} \Rightarrow$
    
    Generate $N$ samples
    
    If a sample direction from $p_1$ or $p_2$ does not hit the light source, it contributes to $\int_{\Omega_{\text{indirect}}}$
    
    If a sample direction from $p_1$ or $p_2$ does not hit the light source, it contributes to $\int_{\Omega_{\text{indirect}}}$
    
    $F = F + \frac{w_i(\omega_i)}{p(I_i)p_{1}(\omega_i)} f_r(p, \omega_i \leftrightarrow \omega_o) L(p \leftarrow \omega_i) \cos(\omega_i, n_p)$
    
    $I_i \in \{3\} \Rightarrow F = F + \frac{w_3(\omega(x_i))}{p(3)p_3(x_i)} f_r(p, \omega_i \leftrightarrow \omega_o) L_e(x_i \rightarrow -\omega_i) G(p, x_i)$
    
    $F = \frac{1}{N} F$ A sample position from $p_3$ contributes to $\int_{\Omega_{\text{direct}}}$ . . . If not, then $V=G=0$.  

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Combination of $\Omega_{\text{direct}}$ and $\Omega_{\text{indirect}}$

- MIS weights
  - E.g. $\omega \in \Omega_{\text{indirect}} \Rightarrow w_1(\omega) = w_2(\omega) = 0.5$
  - $\omega \in \Omega_{\text{direct}} \Rightarrow w_1(\omega) = w_2(\omega) = 0$
  - $\omega(x) \in \Omega_{\text{direct}} \Rightarrow w_3(\omega(x)) = 1$
  - $\omega(x) \in \Omega_{\text{indirect}} \Rightarrow w_3(\omega(x)) = 0$

$F = 0$

Select $I_i \in \{1, 2, 3\}$ from $p$

$I_i \in \{1, 2\} \land \omega_i \notin \Omega_{\text{direct}} \Rightarrow$

$F = F + \frac{w_{I_i}(\omega_i)}{p(I_i)p_{I_i}(\omega_i)} f_r(p, \omega_i \leftrightarrow \omega_o) L(p \leftarrow \omega_i) \cos(\omega_i, n_p)$

$I_i \in \{3\} \Rightarrow E = E + \frac{1}{p(3)p_3(x_i)} f_r(p, \omega_i \leftrightarrow \omega_o) L_e(x_i \rightarrow -\omega_i) G(p, x_i)$

$F = \frac{1}{N} F$
Combination of $\Omega_{\text{direct}}$ and $\Omega_{\text{indirect}}$
Combination of $\Omega_{\text{direct}}$ and $\Omega_{\text{indirect}}$

- If a ray in direction $\omega$ does not hit a light source: $\omega \in \Omega_{\text{indirect}}$
- If a ray in direction $\omega$ hits a light source: $\omega \in \Omega_{\text{direct}}$
- If a light source position $x$ is visible from a surface point $p$, the respective direction $\omega(x)$ is in $\Omega_{\text{direct}}$
- If a light source position $x$ is not visible from a surface point $p$, the respective direction $\omega(x)$ is in $\Omega_{\text{indirect}}$
Combination of $\Omega_{\text{direct}}$ and $\Omega_{\text{indirect}}$

Multiple importance sampling for mixed material and many small light sources
Combination of $\Omega_{\text{direct}}$ and $\Omega_{\text{indirect}}$

- Three PDFs
  - BRDF PDFs $p_1(\omega_i) \propto \cos(\omega_i, n_p)$, $p_2(\omega_i) \propto \cos(r(n_p, \omega_i), \omega_o)^e$
  - $k$ light sources $p_3(I_k) = \frac{1}{k}$, $p_{I_k+3}(x_i) = \frac{1}{A_{I_k}}$ $I_k \in \{1, \ldots, k\}$
  - Discrete PDF for PDF selection, e.g. $p(1) = p(2) = p(3) = \frac{1}{3}$

$$F = 0$$

Select $I_i \in \{1, 2, 3\}$ from $p$

If a sample direction from $p_1$ or $p_2$ does not hit the light source, it contributes to $\int_{\Omega_{\text{indirect}}} \ldots$

Generate $N$ samples

If a sample direction from $p_1$ or $p_2$ does not hit the light source, it contributes to $\int_{\Omega_{\text{indirect}}} \ldots$

$$F = \frac{1}{N} F$$

A sample position from $p_3$ contributes to $\int_{\Omega_{\text{direct}}} \ldots$$$

$$I_i \in \{1, 2\} \land \omega_i \in \Omega_{\text{indirect}} \Rightarrow$$

$$F = F + \frac{w_{I_i}(\omega_i)}{p(I_i)p_{I_i}(\omega_i)} f_r(p, \omega_i \leftrightarrow \omega_o) L(p \leftarrow \omega_i) \cos(\omega_i, n_p)$$

$$I_i \in \{3\} \Rightarrow F = F + \frac{w_3(\omega(x_i))}{p(3)p_3(I_k)p_{I_k+3}(x_i)} f_r(p, \omega_i \leftrightarrow \omega_o) L_e(x_i \rightarrow -\omega_i) G(p, x_i)$$

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Computation of $\int_{\Omega_{\text{indirect}}}$

- Notation: $L^1 = \int_{\Omega^1_d} f_r^1 \cos^1 L^1_e + \int_{\Omega^1_i} f_r^1 \cos^1 L^2$
  
  $L^2 = \int_{\Omega^2_d} f_r^2 \cos^2 L^2_e + \int_{\Omega^2_i} f_r^2 \cos^2 L^3$
  
  $L^3 = \ldots$
Recursive Formulation

\[ L^1 = \int_{\Omega_1^1} f_r^1 \cos^1 L^1_e + \int_{\Omega_1^1} f_r^1 \cos^1 L^2 \]

\[ L^1 = \int_{\Omega_1^1} f_r^1 \cos^1 L^1_e + \int_{\Omega_1^1} f_r^1 \cos^1 \left( \int_{\Omega_1^2} f_r^2 \cos^2 L^2_e + \int_{\Omega_1^2} f_r^2 \cos^2 L^3 \right) \]

\[ L^1 = \int_{\Omega_1^1} f_r^1 \cos^1 L^1_e + \int_{\Omega_1^1} f_r^1 \cos^1 \left( \int_{\Omega_1^2} f_r^2 \cos^2 L^2_e + \int_{\Omega_1^2} f_r^2 \cos^2 \left( \int_{\Omega_1^3} f_r^3 \cos^3 L^3_e + \int_{\Omega_1^3} f_r^3 \cos^3 L^4 \right) \right) \]

- Recursion is terminated by setting \( \int_{\Omega_1^k} = 0 \), e.g.

\[ L^1 = \int_{\Omega_1^1} f_r^1 \cos^1 L^1_e + \int_{\Omega_1^1} f_r^1 \cos^1 \left( \int_{\Omega_1^2} f_r^2 \cos^2 L^2_e + \int_{\Omega_1^2} f_r^2 \cos^2 \left( \int_{\Omega_1^3} f_r^3 \cos^3 L^3_e \right) \right) \]

\[ L^1 = \int_{\Omega_1^1} f_r^1 \cos^1 L^1_e + \int_{\Omega_1^1} f_r^1 \cos^1 \int_{\Omega_1^2} f_r^2 \cos^2 L^2_e + \int_{\Omega_1^1} f_r^1 \cos^1 \int_{\Omega_1^2} f_r^2 \cos^2 \int_{\Omega_1^3} f_r^3 \cos^3 L^3_e + \ldots \]

<table>
<thead>
<tr>
<th>Emitted light after one bounce</th>
<th>Emitted light after two bounces</th>
<th>Emitted light after three bounces</th>
<th>Emitted light after more bounces</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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</tbody>
</table>
Path Tracing

\[ L^1 = \int_{\Omega^1} f_r^1 \cos^1 L_e + \int_{\Omega^1} f_r^1 \cos^1 \int_{\Omega^2} f_r^2 \cos^2 L_e + \int_{\Omega^1} f_r^1 \cos^1 \int_{\Omega^3} f_r^3 \cos^3 L_e + \ldots \]

\[ L^1 \approx \frac{1}{N_{d,1}} \sum_{i=1}^{N_{d,1}} \frac{f_r^1 \cos^1 L_e}{\text{pdf}_d^1} + \frac{1}{N_{i,1}} \sum_{i=1}^{N_{i,1}} \frac{f_r^1 \cos^1}{\text{pdf}_i^1} \left( \frac{1}{N_{d,2}} \sum_{i=1}^{N_{d,2}} \frac{f_r^2 \cos^2}{\text{pdf}_d^2} L_e \right) + \ldots \]

- Taking one sample everywhere

\[ L^1 \approx \frac{f_r^1 \cos^1}{\text{pdf}_d^1} L_e + \frac{f_r^1 \cos^1}{\text{pdf}_i^1} \frac{f_r^2 \cos^2}{\text{pdf}_d^2} L_e + \frac{f_r^1 \cos^1}{\text{pdf}_i^1} \frac{f_r^2 \cos^2}{\text{pdf}_i^2} \frac{f_r^3 \cos^3}{\text{pdf}_d^3} L_e + \ldots \]

Path tracing with next event estimation
Path Tracing with Next Event Estimation

\[ L^1 \approx \frac{f_r^1 \cos^1}{pdf^1_d} L_e^1 + \frac{f_r^1 \cos^1}{pdf^1_i} \frac{f_r^2 \cos^2}{pdf^2_d} L_e^2 + \frac{f_r^1 \cos^1}{pdf^1_i} \frac{f_r^2 \cos^2}{pdf^2_i} \frac{f_r^3 \cos^3}{pdf^3_d} L_e^3 + \ldots \]

Path 1
Path 2
Path 3

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General Path Tracing

\[ L^1 = 0 \]

Generate a sample towards \( p_1 \)

\[ \beta = 1 \quad L_e > 0 \Rightarrow L^1 = L^1 + \beta L_e \]

Generate a sample from \( p_1 \) into the entire hemisphere with pdf\(_1\) towards \( p_2 \)

\[ \beta = \beta \cdot \frac{f^1_r \cos^1}{\text{pdf}^1} \quad L_e > 0 \Rightarrow L^1 = L^1 + \beta L_e \]

Generate a sample from \( p_2 \) into the entire hemisphere with pdf\(_2\) towards \( p_3 \)

\[ \beta = \beta \cdot \frac{f^2_r \cos^2}{\text{pdf}^2} \quad L_e > 0 \Rightarrow L^1 = L^1 + \beta L_e \]

Generate a sample towards \( p_4 \)

\[ \beta = \beta \cdot \frac{f^3_r \cos^3}{\text{pdf}^3} \quad L_e > 0 \Rightarrow L^1 = L^1 + \beta L_e \]

Terminate, if e.g. throughput \( \beta \) smaller than user-defined threshold

Assumes reflective light sources.
General Path Tracing

1 path per pixel.

$n \cdot n \cdot n$ paths per pixel. Sample distribution at second bounce converges to $pdf^2$.

$n$ paths per pixel. Sample distribution at first bounce converges to $pdf^1$.

$n \cdot n \cdot n$ paths per pixel. Sample distribution at third bounce converges to $pdf^3$. 
Jittered Path Sampling per Pixel

- Various strategies
  - Random
  - Stratified
  - Quasi random
    (low-discrepancy sequences)
- Regular sampling would cause aliasing

$n$ samples / paths per pixel.
Image Reconstruction

- Convolution with a normalized kernel function / filter $W$

$$L(x_i) = \int_A L(x')W(\|x_i - x'\|)dx'$$

$$L(x_i) \approx \sum_j L(x_j)W(\|x_i - x_j\|)A(x_j)$$

- E.g., box filter: $W = \text{const}$

- $x_j$ - uniform samples in one pixel $i$

- $A = \frac{\text{area of pixel}}{N} \sum_j W = \frac{1}{\text{area of pixel}}$

- $\sum_j W \cdot A = \frac{\text{area of pixel}}{N} \cdot \frac{1}{\text{area of pixel}}$

- $L(x_i) = \sum_j L(x_j) \cdot W \cdot A = \frac{1}{N} \sum_j L(x_j)$

9 pixels with $N$ samples. Pixel $i$ with representative position $x_i$ and samples / paths at positions $x_j$. 
Path Tracing – Maximum Path Length

- Fixed, user-defined
- Adaptive with Russian Roulette
  - Minimum fixed length, user-defined
  - Termination probability $q$ for additional segments
- Random sample $\xi$:
  $$ F' = \begin{cases} \frac{F - q c}{1 - q} & \xi > q \\ c & \text{otherwise} \end{cases} $$
  Typically $c=0$

- Intuition
  - Some samples are discarded
  - Remaining samples are amplified to account for missing contributions

$$ E[F'] = (1 - q) \frac{E[F] - q c}{1 - q} + q c = E[F] $$
Russian Roulette - Motivation

Fixed path length
- Biased estimator
- Always too small / dark, but consistent
- Converges to correct result
- Completely misses effects that require longer paths
- Example: All paths with zero contribution

Adaptive path length with Russian Roulette
- Unbiased estimator
- Converges to correct result
- Arbitrarily long paths potentially capture more effects than fixed path lengths, although with low quality
- Example: Most samples with zero contribution
  Some very long paths with non-zero contr.
Splitting

– Adaptive sample counts at intersections

Without splitting

Splitting

$n$ samples

$k$ samples

$n$ samples

1 sample
Path Tracing – Degrees of Freedom

- \[ L = \prod_{i=1}^{N} \frac{f_r^i \cos^i}{\text{pdf}^i} L_e^N \]
- Path length, e.g. Russian Roulette
- PDFs at each intersection, e.g. MIS with light and material sampling
- Next event estimation, i.e. light sampling at each intersection
- Number of samples at each intersection, i.e. splitting
Current Variants

– Bidirectional path generation
  – Motivation: Samples from the light source into the scene are as important as samples from the sensor into the scene
  – Symmetric setting
– Metropolis sampling
  – Path mutations instead of random sampling
  – Small mutations in case of relevant paths
  – Large mutations in case of less relevant paths