Advanced Computer Graphics

Rendering Equation

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Outline

- rendering equation
- Monte Carlo integration
- sampling of random variables
Reflection andRendering Equation

- reflection equation at point $p$ for reflective surfaces
  - $L_o(p, \omega_o) = \int_{2\pi} f_r(p, \omega_i, \omega_o) L_i(p, \omega_i) \cos \theta_i d\omega_i$
  - incident radiance - weighted with the BRDF - is integrated over the hemisphere to compute the outgoing radiance
  - expresses energy balance between surfaces
  - outgoing radiance from a surface can be incident to another surface

- rendering equation at point $p$ for reflective surfaces
  - $L_o(p, \omega_o) = L_e(p, \omega_o) + \int_{2\pi} f_r(p, \omega_i, \omega_o) L_i(p, \omega_i) \cos \theta_i d\omega_i$
  - adds emissive surfaces to the reflection equation
  - exitant radiance is the sum of emitted and reflected radiance
  - expresses the steady state of radiance in a scene including light sources
Ray-Casting Operator

- In general, the incoming radiance is not only determined by light sources, but also by outgoing radiance of reflective surfaces.
- Incident radiance $L_i(p, \omega_i)$ can be computed by tracing a ray from $p$ into direction $\omega_i$.
- Ray-casting operator $r_c(p, \omega_i)$
  - Nearest hit point from $p$ into direction $\omega_i$.
  - $L_i(p, \omega_i) = L_o(r_c(p, \omega_i), -\omega_i)$
  - If no surface is hit, radiance from a background or light source can be returned.

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Rendering Equation with Ray-Casting Operator

- using the ray-casting operator,
  \[ L_o(p, \omega_o) = L_e(p, \omega_o) + \int_{2\pi} f_r(p, \omega_i, \omega_o) L_i(p, \omega_i) \cos \theta_i d\omega_i \]
  can be rewritten as
  \[ L_o(p, \omega_o) = L_e(p, \omega_o) + \int_{2\pi} f_r(p, \omega_i, \omega_o) L_o(r_c(p, \omega_i), -\omega_i) \cos \theta_i d\omega_i \]
- goal: computation of the outgoing radiance \( L_o(p, \omega_o) \) at all points \( p \) into all directions \( \omega_o \)
  - towards the camera to compute the image
  - towards other surface points to account for indirect illumination
Forms of the Rendering Equation

- **hemisphere form**
  \[
  L_o(p, \omega_o) = L_e(p, \omega_o) + \int_{2\pi} f_r(p, \omega_i, \omega_o) L_o(r_c(p, \omega_i), -\omega_i) \cos \theta_i d\omega_i
  \]

- **area form**
  - \(p\) is a sample point on a surface \(dA\)
  - visibility function
    \[
    \forall(p, p') : V(p, p') = \begin{cases} 
    1 & \text{if } p \text{ and } p' \text{ see each other} \\
    0 & \text{if } p \text{ and } p' \text{ do not see each other}
    \end{cases}
    \]
  - solid angle vs. area \(d\omega_i = \frac{\cos \theta' dA}{\|p' - p\|^2}\)
  - \(\cos \theta' = n' \cdot -\omega_i\)
  \[
  L_o(p, \omega_o) = L_e(p, \omega_o) + \\
  \int_A f_r(p, \omega_i, \omega_o) L_o(p', -\omega_i) \frac{\cos \theta_i \cos \theta'}{|p' - p|^2} V(p, p') dA
  \]
Forms of the Rendering Equation

- the area form works with a visibility term
  - useful for direct illumination from area lights
- the hemisphere form works with the ray-casting operator
  - useful for indirect illumination

hémisphère form

area form
Solving the Rendering Equation

- recursively cast rays into the scene
- maximum recursion depth due to absorption of light
- for point lights, directional lights, perfect reflection and transmission, the integrals reduce to simple sums
  - radiance from only a few directions contributes to the outgoing radiance
- for area lights and indirect illumination, i.e. diffuse-diffuse light transport, Monte Carlo techniques are used to numerically evaluate the multi-dimensional integrals
Outline

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- Monte Carlo integration
- sampling of random variables
Introduction

- approximately evaluate the integral
  \[ \int_{2\pi} \ f_r(p, \omega_i, \omega_o) L_o(r_c(p, \omega_i), -\omega_i) \cos \theta_i \, d\omega_i \]
  by
  - randomly sampling the hemisphere
  - tracing rays into the sample directions
  - computing the incoming radiance from the sample directions

- challenge
  - approximate the integral as exact as possible
  - trace as few rays as possible
  - trace relevant rays
    - for diffuse surfaces, rays in normal direction are more relevant than rays perpendicular to the normal
    - for specular surfaces, rays in reflection direction are relevant
    - rays to light sources are relevant
Properties

- **benefits**
  - processes only evaluations of the integrand at arbitrary points in the domain
  - works for a large variety of integrands, e.g., it handles discontinuities
  - appropriate for integrals of arbitrary dimensions

- **drawbacks**
  - using $n$ samples, the scheme converges to the correct result with $O(n^{\frac{1}{2}})$, i.e. to half the error, $4n$ samples are required
  - errors are perceived as noise, i.e. pixels are arbitrarily too bright or dark
  - evaluation of the integrand at a point is expensive
Continuous Random Variables

- continuous random variables $X$ (in contrast to discrete random variable)
- canonical uniform random variable $0 \leq \xi < 1$
  - samples from arbitrary distributions can be computed from $\xi$
- probability density function (PDF) $p(x)$
  - the probability of a random variable taking certain value ranges
    - $Pr(x_0 \leq X \leq x_1) = \int_{x_0}^{x_1} p(x) \, dx$
    - $p(x) \geq 0 \quad \forall x \in [a, b]$
    - $\int_{a}^{b} p(x) \, dx = 1$
- cumulative distribution function (CDF) $P(x)$
  - describes the probability of a random variable to be less or equal to $x$
    - $Pr(X \leq x) = P(x)$
    - $Pr(x_0 \leq X \leq x_1) = P(x_1) - P(x_0)$
    - $0 \leq P(x) \leq 1$
Expected Value

- motivation: expected value of an estimator function is equal to the integral in the rendering equation
- expected value $E_p[f(x)]$ of a function $f(x)$ is defined as the weighted average value of the function over a domain $D$
  \[
  E_p[f(x)] = \int_D f(x) \, p(x) \, dx \quad \text{with} \quad \int_D p(x) \, dx = 1
  \]
- properties
  - $E[a \, f(x)] = a \, E[f(x)]$
  - $E[\sum_i f(X_i)] = \sum_i E[f(X_i)]$ for independent random variables $X_i$
- example for uniform $p$
  \[
  E_p[\cos(x)] = \int_0^\pi \cos(x) \frac{1}{\pi} \, dx = \frac{1}{\pi} (-\sin \pi + \sin 0) = 0
  \]
motivation: quantifies the error of a Monte Carlo algorithm

variance $V$ of a function is the expected deviation of the function from its expected value

$$V[f(x)] = E[(f(x) - E[f(x)])^2]$$

properties

- $V[af(x)] = a^2V[f(x)]$
- $V[f(x)] = E[(f(x))^2] - E[f(x)]^2$
- $\sum_i V[f(X_i)] = V[\sum_i f(X_i)]$ for independent random variables $X_i$
Monte Carlo Estimator

Uniform Random Variables

- motivation: approximation of the integral in the rendering equation
- goal: computation of $\int_a^b f(x)\,dx$
- uniformly distributed random variables $X_i \in [a, b]$
- probability density function $p(x) = \frac{1}{b-a}$ constant and integration to one
- Monte Carlo estimator $F_N = \frac{b-a}{N} \sum_{i=1}^{N} f(X_i)$
- expected value of $F_N$ is equal to the integral $\int_a^b f(x)\,dx$
  - $E[F_N] = \int_a^b f(x)\,dx$
- variance $V = \frac{1}{N-1} \sum_{i=1}^{N} [f(X_i) - E[F_N]]^2$
  - convergence rate of $O(\sqrt{N})$
  - independent from the dimensionality
  - appropriate for high-dimensional integrals
Monte Carlo Estimator

Uniform Random Variables

\[ E[F_N] = E \left[ \frac{b-a}{N} \sum_{i=1}^{N} f(X_i) \right] \]

\[ = \frac{b-a}{N} \sum_{i=1}^{N} E[f(X_i)] \]

\[ = \frac{b-a}{N} \sum_{i=1}^{N} \int_{a}^{b} f(x)p(x)dx \]

\[ = \frac{b-a}{N} \sum_{i=1}^{N} \int_{a}^{b} f(x) \frac{1}{b-a}dx \]

\[ = \frac{1}{N} \sum_{i=1}^{N} \int_{a}^{b} f(x)dx \]

\[ = \int_{a}^{b} f(x)dx \]
Examples - Uniform Random Variables

- integral $\int_0^1 5x^4 \, dx = 1$
- estimator $F_N = \frac{1}{N} \sum_{i=1}^N 5X_i^4$
- for an increasing number of uniformly distributed random variables $X_i$, the estimator converges to one

$F_N = \frac{b-a}{N} \sum_{i=1}^N f(X_i)$
$F_N = (b-a) \frac{1}{N} \sum_{i=1}^N f(X_i)$
$F_N = (b-a) \overline{f(x)}$
$E[F_N] = \int_a^b f(x) \, dx$

uniformly distributed random samples

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Monte Carlo Estimator

Non-uniform Random Variables

- Monte Carlo estimator \( F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)} \)  
  \( p(X_i) \neq 0 \)

\[
E[F_N] = E \left[ \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)} \right] \\
= \frac{1}{N} \sum_{i=1}^{N} \int_{a}^{b} \frac{f(x)}{p(x)} p(x) dx \\
= \frac{1}{N} \sum_{i=1}^{N} \int_{a}^{b} f(x) dx \\
= \int_{a}^{b} f(x) dx
\]
Monte Carlo Estimator

Multiple Dimensions

- samples $X_i$ are multidimensional
- e.g. $\int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} f(x, y, z) dx dy dz$
- uniformly distributed random samples
  $(x_0, y_0, z_0) \leq X_i = (x_i, y_i, z_i) \leq (x_1, y_1, z_1)$
- probability density function $p(X_i) = \frac{1}{x_1-x_0} \frac{1}{y_1-y_0} \frac{1}{z_1-z_0}$
- Monte Carlo estimator
  $F_N = \frac{(x_1-x_0)(y_1-y_0)(z_1-z_0)}{N} \sum_{i=1}^{N} f(X_i)$
- $N$ can be arbitrary, $N$ is independent from the dimensionality
Monte Carlo Estimator
Integration over a Hemisphere

- approximate computation of the irradiance at a point
  \[ E_i(p) = \int_{2\pi} L_i(p, \omega) \cos \theta d\omega \]
  \[ = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} L_i(p, \theta, \phi) \cos \theta \sin \theta d\theta d\phi \]

- estimator
  \[ F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)} = \frac{1}{N} \sum_{i=1}^{N} \frac{L_i(p, \theta, \phi) \cos \theta \sin \theta}{p(\theta, \phi)} \]

- probability distribution
  - should be similar to the shape of the integrand
  - as incident radiance is weighted with \( \cos \theta \), it is appropriate to generate more samples close to the top of the hemisphere
  - \( p(\omega) \propto \cos \theta \)
Monte Carlo Estimator
Integration over a Hemisphere

- Probability distribution (cont.)
  \[
  \int_{2\pi}^{2\pi} c \ p(\omega) d\omega = 1 \\
  \int_{0}^{\pi} \int_{0}^{\frac{\pi}{2}} c \ \cos \theta \sin \theta \ d\theta \ d\phi = 1 \\
  c \ \frac{2\pi}{1+1} = 1 \\
  c = \frac{1}{\pi} \\
  p(\theta, \phi) = \frac{\cos \theta \sin \theta}{\pi}
  \]
- Estimator
  \[
  F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{L_i(p, \theta, \phi) \cos \theta \sin \theta}{p(\theta, \phi)} \\
  = \frac{\pi}{N} \sum_{i=1}^{N} L_i(p, \theta, \phi)
  \]
Monte Carlo Integration

Steps

- choose an appropriate probability density function
- generate random samples according to the PDF
- evaluate the function for all samples
- average the weighted function values
Monte Carlo Estimator

Error

- Variance
  \[ V = \frac{1}{N} \int_a^b \left( \frac{f(x)}{p(x)} - F_N \right)^2 p(x) \, dx \]
- Estimator
  \[ V_N = \frac{1}{N-1} \sum_{i=1}^{N} \left[ f(X_i) - F_N \right]^2 \]
- For increasing \( N \)
  - The variance decreases with \( O(N) \)
  - The standard deviation decreases with \( O(N^{\frac{1}{2}}) \)
- Variance is perceived as noise
Monte Carlo Estimator

Variance Reduction / Error Reduction

- **importance sampling**
  - motivation: contributions of larger function values are more important
  - PDF should be similar to the shape of the function
  - optimal PDF $p(x) = \frac{f(x)}{\int f(x) dx}$
  - e.g., if incident radiance is weighted with $\cos \theta$, the PDF should choose more samples close to the normal direction

- **stratified sampling**
  - domain subdivision into strata does not increase the variance

- **multi-jittered sampling**
  - alternative to random samples for, e.g., uniform sampling of area lights
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  - inversion method
  - rejection method
  - transforming between distributions
  - 2D sampling
  - examples
Inversion Method

- mapping of a uniform random variable to a goal distribution
- discrete example
  - four outcomes with probabilities $p_1, p_2, p_3, p_4$ and $\sum_i p_i = 1$
- computation of the cumulative distribution function $P(i) = \sum_{j=1}^{i} p_j$
Inversion Method

- discrete example cont.
  - take a uniform random variable $\xi$
  - $P^{-1}(\xi)$ has the desired distribution

- continuous case
  - $P$ and $P^{-1}$ are continuous functions
  - start with the desired PDF $p(x)$
  - compute $P(x) = \int_0^x p(x')dx'$
  - compute the inverse $P^{-1}(x)$
  - obtain a uniformly distributed variable
  - compute $X_i = P^{-1}(\xi)$ which adheres to $p(x)$
Inversion Method

Example 1

- power distribution \( p(x) \propto x^n \)
  - e.g., for sampling the Blinn microfacet model
- computation of the PDF
  - \( \int_0^1 c x^n \, dx = 1 \Rightarrow c \frac{x^{n+1}}{n+1} \bigg|_0^1 = 1 \Rightarrow c = n + 1 \)
- PDF \( p(x) = (n + 1)x^n \)
- CDF \( P(x) = \int_0^x p(x') \, dx' = x^{n+1} \)
- inverse of the CDF \( P^{-1}(x) = \frac{n+1}{\sqrt{x}} \)
- sample generation
  - generate uniform random samples \( 0 \leq \xi \leq 1 \)
  - \( X = \frac{n+1}{\sqrt{\xi}} \) are samples from the power distribution \( p(x) = (n + 1)x^n \)
Inversion Method

Example 2

- exponential distribution $p(x) \propto e^{-ax}$
  - e.g., for considering participating media
- computation of the PDF
  - $\int_0^\infty c \ e^{-ax} \, dx = -\frac{c}{a} \ e^{-ax}\big|_0^\infty = \frac{c}{a} = 1$
  
  $p(x) = a \ e^{-ax}$
- CDF $P(x) = \int_0^x p(x') \, dx' = 1 - e^{-ax}$
- inverse of the CDF $P^{-1}(x) = -\frac{\ln(1-x)}{a}$
- sample generation
  - generate uniform random samples $0 \leq \xi \leq 1$
  - $X = -\frac{\ln(1-\xi)}{a}$ are samples from the power distribution $p(x) = a \ e^{-ax}$
Inversion Method

Example 3

- piecewise-constant distribution
  - e.g., for environment lighting

\[
f(x) = \begin{cases} 
  v_0 & x_0 \leq x < x_1 \\
  v_1 & x_1 \leq x < x_2 \\
  \vdots & \vdots \\
  x_i &= \Delta \cdot i \\
\end{cases}
\]

\[\Delta = \frac{1}{N}\]

- PDF \[p(x) = \frac{1}{c} f(x)\]

with \[c = \int_0^1 f(x) \, dx = \sum_{i=0}^{N-1} \Delta \cdot v_i = \sum_{i=0}^{N-1} \frac{v_i}{N}\]
Inversion Method

Example 3

- **CDF**
  \[ P(x_0) = 0 \]
  \[ P(x_1) = \int_{x_0}^{x_1} p(x)dx = \Delta \cdot \frac{v_0}{c} = \frac{v_0}{Nc} = P(x_0) + \frac{v_0}{Nc} \]
  \[ P(x_2) = \int_{x_0}^{x_2} p(x)dx = \int_{x_0}^{x_1} p(x)dx + \int_{x_1}^{x_2} p(x)dx = P(x_1) + \frac{v_1}{Nc} \]
  \[ P(x_i) = P(x_{i-1}) + \frac{v_{i-1}}{Nc} \]

- **CDF is linear between** \( x_i \) **and** \( x_{i+1} \) **with slope** \( \frac{v_i}{c} \)

- **Sample generation**
  - generate uniform random samples \( 0 \leq \xi \leq 1 \)
  - compute \( x_i \) with \( P(x_i) \leq \xi \) and \( \xi < P(x_{i+1}) \)
  - compute \( d \) with \( P(x_i) + d(P(x_{i+1}) - P(x_i)) = \xi \)
  - \( X = x_i + d(x_{i+1} - x_i) = x_i + \frac{d}{N} \) are samples from \( p(x) = \frac{1}{c} f(x) \)
Outline

- rendering equation
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  - inversion method
  - rejection method
  - transforming between distributions
- 2D sampling
- examples
Rejection Method

- draws samples according to a function $f(x)$
  - dart-throwing approach
  - works with a PDF $p(x)$ and a scalar $c$ with $f(x) < c \cdot p(x)$
- properties
  - $f(x)$ is not necessarily a PDF
  - PDF, CDF and inverse CDF do not have to be computed
  - simple to implement
  - useful for debugging purposes
- sample generation
  - generate a uniform random sample $0 \leq \xi < 1$
  - generate a sample $X$ according to $p(x)$
  - accept $X$ if $\xi \cdot c \cdot p(X) \leq f(X)$
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Transforming Between Distributions

- computation of a resulting PDF, when a function is applied to samples from an arbitrary distribution
  - random variables $X_i$ are drawn from $p_x(x)$
  - bijective transformation (one-to-one mapping) $Y_i = y(X_i)$
  - How does the distribution $p_y(y)$ look like?

\[
Pr\{Y \leq y(x)\} = Pr\{X \leq x\}
\]
\[
P_y(y) = P_y(y(x)) = P_x(x)
\]
\[
p_y(y) = \frac{p_x(x)}{|y'(x)|}
\]

- example $p_x(x) = 2x \quad 0 \leq x \leq 1$
  - $y(x) = \sin x \quad x(y) = \arcsin y$
  - $y'(x) = \cos x$
  - $p_y(y) = \frac{p_x(x)}{|\cos x|} = \frac{2 \arcsin y}{\sqrt{1-y^2}}$
Transforming Between Distributions

- multiple dimensions
  - $X_i$ is an n-dimensional random variable
  - $Y_i = T(X_i)$ is a bijective transformation

- transformation of the PDF
  \[ p_y(y) = \frac{p_x(x)}{|J_T(x)|} \]
  \[ J_T(x) = \begin{pmatrix}
    \frac{\partial T_1}{\partial x_1} & \cdots & \frac{\partial T_1}{\partial x_n} \\
    \vdots & \ddots & \vdots \\
    \frac{\partial T_n}{\partial x_1} & \cdots & \frac{\partial T_n}{\partial x_n}
  \end{pmatrix} \]

- example (polar coordinates)
  - samples $(r, \theta)$ with density $p(r, \theta)$
  - corresponding density $p(x, y)$ with $x = r \cos \theta$ and $y = r \sin \theta$
  \[ J_T(x) = \begin{pmatrix}
    \cos \theta & -r \sin \theta \\
    \sin \theta & \ r \cos \theta
  \end{pmatrix} \quad |J_T(x)| = r(\cos^2 \theta + \sin^2 \theta) = r \]
  \[ p(x, y) = \frac{1}{r} p(r, \theta) \quad p(r, \theta) = r \cdot p(x, y) \]
Transforming Between Distributions

- example (spherical coordinates)
  - $x = r \sin \theta \cos \phi$
  - $y = r \sin \theta \sin \phi$
  - $z = r \cos \theta$
  - $p(r, \theta, \phi) = r^2 \sin \theta \cdot p(x, y, z)$

- example (solid angle)
  - $Pr\{\omega \in \Omega\} = \int_{\Omega} p(\omega) d\omega$
  - $d\omega = \sin \theta \ d\theta \ d\phi$
  - $p(\theta, \phi) = \sin \theta \cdot p(\omega)$
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Concept

generation of samples from a 2D joint density function $p(x, y)$

general case

- compute the marginal density function $p_x(x) = \int p(x, y) dy$
- compute the conditional density function $p_y(y|x) = \frac{p(x,y)}{p_x(x)}$
- generate a sample $X$ according to $p_x(x)$
- generate a sample $Y$ according to $p_y(y|X) = \frac{p(x,y)}{p_x(x)}$

marginal density function

- integral of $p(x, y)$ for a particular $x$ over all $y$-values

conditional density function

- density function for $y$ given a particular $x$
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Uniform Sampling of a Hemisphere

- PDF is constant with respect to a solid angle $p(\omega) = c$
  \[ \int_{2\pi} p(\omega) \, d\omega = 1 \Rightarrow c \int_{2\pi} \, d\omega = 1 \Rightarrow c = \frac{1}{2\pi} \]
- $p(\omega) = \frac{1}{2\pi} \Rightarrow p(\theta, \phi) = \frac{\sin \theta}{2\pi}$
- marginal density function
  - $p_\theta(\theta) = \int_0^{2\pi} p(\theta, \phi) \, d\phi = \int_0^{2\pi} \frac{\sin \theta}{2\pi} \, d\phi = \sin \theta$
- conditional density for $\phi$
  - $p_\phi(\phi | \theta) = \frac{p(\theta, \phi)}{p_\theta(\theta)} = \frac{1}{2\pi}$
- inversion method
  - $P_\theta(\theta) = \int_0^\theta \sin \theta' \, d\theta' = -\cos \theta + 1$
  - $P_\phi(\phi | \theta) = \int_0^\phi \frac{1}{2\pi} \, d\phi' = \frac{\phi}{2\pi}$
Uniform Sampling of a Hemisphere

- inversion method cont.
  - inverse functions of the cumulative distribution functions
    - $\theta = \arccos(1 - \xi_1)$
    - $\phi = 2\pi \xi_2$
  - generating uniformly sampled random values $\xi_1$ and $\xi_2$
  - applying the inverse CDFs to obtain $\theta$ and $\phi$

- conversion to Cartesian space
  - $x = \sin \theta \cos \phi = \cos(2\pi \xi_2) \sqrt{1 - (1 - \xi_1)^2}$
  - $y = \sin \theta \sin \phi = \sin(2\pi \xi_2) \sqrt{1 - (1 - \xi_1)^2}$
  - $z = \cos \theta = 1 - \xi_1$

- $(x, y, z)^T$ is a normalized direction
Uniform Sampling of a Hemisphere

- illustration for $\theta$

$$\theta = \arccos(1 - \xi_1)$$

generate less samples for smaller angles $\theta$
**Uniform Sampling of a Unit Disk**

- PDF is constant with respect to area: \( p(x, y) = \frac{1}{\pi} \)
- \( p(r, \theta) = r \cdot p(x, y) \Rightarrow \frac{r}{\pi} \)
- marginal density function
  - \( p_r(r) = \int_0^{2\pi} p(r, \theta) d\theta = 2r \)
- conditional density
  - \( p_\theta(\theta|r) = \frac{p(r, \theta)}{p_r(r)} = \frac{1}{2\pi} \)
- inversion method
  - \( P_r(r) = \int_0^r 2r' dr' = r^2 \)
  - \( P_\theta(\theta|r) = \int_0^{2\pi} \frac{1}{2\pi} d\theta' = \frac{\theta}{2\pi} \)
**Uniform Sampling of a Unit Disk**

- inversion method cont.
  - inverse functions of the cumulative distribution functions
  - $r = \sqrt{\xi_1}$
  - $\theta = 2\pi\xi_2$
  - generating uniformly sampled random values $\xi_1$ and $\xi_2$
  - applying the inverse CDFs to obtain $r$ and $\theta$

![Graph showing the relationship between $\xi_1$ and $r$](image)

*generate less samples for smaller radii*
Uniform Sampling of a Cosine-Weighted Hemisphere

- PDF is proportional to \( \cos \theta \)
  \( p(\omega) \propto \cos \theta \)
  \[
  \int_{2\pi} \, c \, p(\omega) \, d\omega = 1 = \int_0^{2\pi} \int_0^{\pi/2} c \, \cos \theta \sin \theta \, d\theta \, d\phi = c \ 2\pi \int_0^{\pi/2} \cos \theta \sin \theta \, d\theta = c \ 2\pi \frac{1}{2} = 1
  \]
- marginal density function
  \( p(\theta, \phi) = \frac{1}{\pi} \cos \theta \sin \theta \)
  \[
  p_\theta(\theta) = \int_0^{2\pi} p(\theta, \phi) d\phi = \int_0^{2\pi} \frac{1}{\pi} \cos \theta \sin \theta d\phi = 2 \cos \theta \sin \theta
  \]
- conditional density for \( \phi \)
  \( p_\phi(\phi|\theta) = \frac{p(\theta, \phi)}{p_\theta(\theta)} = \frac{1}{2\pi} \)
- inversion method
  \[
  P_\theta(\theta) = \int_0^\theta 2 \cos \theta' \sin \theta' d\theta' = 2 \left[ -\frac{\cos^2 \theta'}{2} \right]_0^\theta = 2 \left( -\frac{\cos^2 \theta}{2} + \frac{1}{2} \right) = \sin^2 \theta
  \]
  \[
  P_\phi(\phi|\theta) = \int_0^\phi \frac{1}{2\pi} d\phi' = \frac{\phi}{2\pi}
  \]
Uniform Sampling of a Cosine-Weighted Hemisphere

- inversion method cont.
  - inverse functions of the cumulative distribution functions
    - \( \theta = \arcsin(\sqrt{\xi_1}) \)
    - \( \phi = 2\pi \xi_2 \)
  - generating uniformly sampled random values \( \xi_1 \) and \( \xi_2 \)
  - applying the inverse CDFs to obtain \( \theta \) and \( \phi \)

- conversion to Cartesian space
  - \( x = \sin \theta \cos \phi = \cos(2\pi \xi_2) \sqrt{1 - \xi_1} \)
  - \( y = \sin \theta \sin \phi = \sin(2\pi \xi_2) \sqrt{1 - \xi_1} \)
  - \( z = \cos \theta = 1 - \xi_1 \)
  - \((x, y, z)^T\) is a normalized direction

\( x \)-y-values uniformly sample a unit disk, i.e., cosine-weighted samples of the hemisphere can also be obtained by uniformly sampling a unit sphere and projecting the samples onto the hemisphere.
Uniform Sampling of a Cosine-Weighted Hemisphere

- illustration for $\theta$

\[ \theta = \arcsin(\xi_1) \]

- generate less samples for smaller and larger angles $\theta$
- cosine-weighted hemisphere (top view, side view)
- uniform hemisphere (top view)
Uniform Sampling of a Triangle

- sampling an isosceles right triangle of area 0.5
  - $u, v$ can be interpreted as Barycentric coordinates
  - can be used to generate samples for arbitrary triangles
- $p(u,v) = 2$
- marginal density function
  - $p_u(u) = \int_0^{1-u} p(u,v) \, dv = 2 \int_0^{1-u} dv = 2(1 - u)$
- conditional density
  - $p_v(v|u) = \frac{p(u,v)}{p_u(u)} = \frac{1}{1-u}$
- inversion method
  - $P_u(u) = \int_0^u 2 - 2u' \, du' = 2u - u^2$
  - $P_v(v|u) = \int_0^v \frac{1}{1-u} \, dv' = \frac{v}{1-u}$
Uniform Sampling of a Triangle

- inversion method cont.
  - inverse functions of the cumulative distribution functions
    - $u = 1 - \sqrt{\xi_1}$
    - $v = \xi_2 \sqrt{\xi_1}$
  - generating uniformly sampled random values $\xi_1$ and $\xi_2$
  - applying the inverse CDFs to obtain $u$ and $v$
Piecewise-Constant 2D Distribution

- $n_u \times n_v$ samples defined over $(u, v) \in [0, 1]^2$
  - e.g., an environment map
- $f(u, v)$ is defined by a set of $n_u \times n_v$ values $f[u_i, v_i]$
  - $u_i \in [0, \ldots, n_u - 1]$ $v_i \in [0, \ldots, n_v - 1]$
  - $f[u_i, v_i]$ is the value of $f(u, v)$ in the range $\left[\frac{i}{n_u}, \frac{i+1}{n_u}\right] \times \left[\frac{j}{n_v}, \frac{j+1}{n_v}\right]$
  - $f(u, v) = f[u_i, v_i]$ with $\tilde{u} = \lfloor n_u u \rfloor$ and $\tilde{v} = \lfloor n_v v \rfloor$

- integral over the domain
  - $I_f = \int \int f(u, v) \, du \, dv = \frac{1}{n_u n_v} \sum_i \sum_j f[u_i, v_j]$
- PDF
  - $p(u, v) = \frac{1}{I_f} f(u, v) = \frac{1}{I_f} f[\tilde{u}, \tilde{v}]$
**Piecewise-Constant 2D Distribution**

- **marginal density function**
  - \( p_v(v) = \int p(u,v) \, du = \frac{1}{I_f} \frac{1}{n_u} \sum_i f[u_i, \tilde{v}] \)
  - piecewise-constant 1D function
  - defined by \( n_v \) values \( p_v[\tilde{v}] \)

- **conditional density**
  - \( p_u(u|v) = \frac{p(u,v)}{p_v(v)} = \frac{1}{I_f} \frac{f[\tilde{u},\tilde{v}]}{p[\tilde{v}]} \)
  - piecewise-constant 1D function

- **sample generation**
  - see example 3 of the inversion method
Piecewise-Constant
2D Distribution

- environment map

- low-resolution of the marginal density function and the conditional distributions for rows
  - first, a row is selected according to the marginal density function
  - then, a column is selected from the row's 1D conditional distribution

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