Advanced Computer Graphics

Rendering Equation

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Outline

- rendering equation
- Monte Carlo integration
- sampling of random variables
Reflection and Rendering Equation

- reflection equation at point $p$ for reflective surfaces
  - $L_o(p, \omega_o) = \int_{2\pi} f_r(p, \omega_i, \omega_o) L_i(p, \omega_i) \cos \theta_i \, d\omega_i$
  - incident radiance - weighted with the BRDF - is integrated over the hemisphere to compute the outgoing radiance
  - expresses energy balance between surfaces
  - outgoing radiance from a surface can be incident to another surface

- rendering equation at point $p$ for reflective surfaces
  - $L_o(p, \omega_o) = L_e(p, \omega_o) + \int_{2\pi} f_r(p, \omega_i, \omega_o) L_i(p, \omega_i) \cos \theta_i \, d\omega_i$
  - adds emissive surfaces to the reflection equation
  - exitant radiance is the sum of emitted and reflected radiance
  - expresses the steady state of radiance in a scene including light sources
Ray-Casting Operator

- in general, the incoming radiance is not only determined by light sources, but also by outgoing radiance of reflective surfaces
- incident radiance $L_i(p, \omega_i)$ can be computed by tracing a ray from $p$ into direction $\omega_i$
- ray-casting operator $r_c(p, \omega_i)$
  - nearest hit point from $p$ into direction $\omega_i$
  - $L_i(p, \omega_i) = L_o(r_c(p, \omega_i), -\omega_i)$
  - if no surface is hit, radiance from a background or light source can be returned
Rendering Equation with Ray-Casting Operator

- using the ray-casting operator,
  $$L_o(p, \omega_o) = L_e(p, \omega_o) + \int_{2\pi} f_r(p, \omega_i, \omega_o) L_i(p, \omega_i) \cos \theta_i d\omega_i$$
  can be rewritten as
  $$L_o(p, \omega_o) = L_e(p, \omega_o) + \int_{2\pi} f_r(p, \omega_i, \omega_o) L_o(r_c(p, \omega_i), -\omega_i) \cos \theta_i d\omega_i$$

- goal: computation of the outgoing radiance $L_o(p, \omega_o)$ at all points $p$ into all directions $\omega_o$
  - towards the camera to compute the image
  - towards other surface points to account for indirect illumination
**Forms of the Rendering Equation**

- **hemisphere form**
  \[ L_o(p, \omega_o) = L_e(p, \omega_o) + \int_{2\pi+} f_r(p, \omega_i, \omega_o) L_o(r_c(p, \omega_i), -\omega_i) \cos \theta_i d\omega_i \]

- **area form**
  - \( p \) is a sample point on a surface \( dA \)
  - visibility function
    \[ \forall(p, p') : V(p, p') = \begin{cases} 1 & \text{if } p \text{ and } p' \text{ see each other} \\ 0 & \text{if } p \text{ and } p' \text{ do not see each other} \end{cases} \]
  - solid angle vs. area
    \[ d\omega_i = \frac{\cos \theta' \, dA}{\|p' - p\|^2} \]
  - \( \cos \theta' = n' \cdot -\omega_i \)
  \[ L_o(p, \omega_o) = L_e(p, \omega_o) + \int_A f_r(p, \omega_i, \omega_o) L_o(p', -\omega_i) \frac{\cos \theta_i \cos \theta'}{\|p' - p\|^2} V(p, p') \, dA \]
Forms of the Rendering Equation

- the area form works with a visibility term
  - useful for direct illumination from area lights
- the hemisphere form works with the ray-casting operator
  - useful for indirect illumination
Solving the Rendering Equation

- recursively cast rays into the scene
- maximum recursion depth due to absorption of light
- for point lights, directional lights, perfect reflection and transmission, the integrals reduce to simple sums
  - radiance from only a few directions contributes to the outgoing radiance
- for area lights and indirect illumination, i.e. diffuse-diffuse light transport, Monte Carlo techniques are used to numerically evaluate the multi-dimensional integrals
Outline

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Introduction

- approximately evaluate the integral
  \[ \int_{2\pi} f_r(p, \omega_i, \omega_o) L_o(r_c(p, \omega_i), -\omega_i) \cos \theta_i d\omega_i \]
  by
  - randomly sampling the hemisphere
  - tracing rays into the sample directions
  - computing the incoming radiance from the sample directions

- challenge
  - approximate the integral as exact as possible
  - trace as few rays as possible
  - trace relevant rays
    - for diffuse surfaces, rays in normal direction are more relevant than rays perpendicular to the normal
    - for specular surfaces, rays in reflection direction are relevant
    - rays to light sources are relevant
Properties

- benefits
  - processes only evaluations of the integrand at arbitrary points in the domain
  - works for a large variety of integrands, e.g., it handles discontinuities
  - appropriate for integrals of arbitrary dimensions

- drawbacks
  - using n samples, the scheme converges to the correct result with $O(n^{1/2})$, i.e. to half the error, 4n samples are required
  - errors are perceived as noise, i.e. pixels are arbitrarily too bright or dark
  - evaluation of the integrand at a point is expensive
Continuous Random Variables

- continuous random variables $X$ (in contrast to discrete random variable)
- canonical uniform random variable $0 \leq \xi < 1$
  - samples from arbitrary distributions can be computed from $\xi$
- probability density function (PDF) $p(x)$
  - the probability of a random variable taking certain value ranges
    $$Pr(x_0 \leq X \leq x_1) = \int_{x_0}^{x_1} p(x) \, dx$$
    - The probability, that the random variable has a certain exact value, is 0.
    - $p(x) \geq 0 \quad \forall x \in [a, b]$
    - $\int_a^b p(x) \, dx = 1$ - The probability, that the random variable is in the specified domain, is 1.
- cumulative distribution function (CDF) $P(x)$
  - describes the probability of a random variable to be less or equal to $x$
    $$Pr(X \leq x) = P(x) \quad Pr(x_0 \leq X \leq x_1) = P(x_1) - P(x_0)$$
  - $0 \leq P(x) \leq 1$
**Expected Value**

- motivation: expected value of an estimator function is equal to the integral in the rendering equation
- expected value $E_p[f(x)]$ of a function $f(x)$ is defined as the weighted average value of the function over a domain $D$
  \[
  E_p[f(x)] = \int_D f(x) \, p(x) \, dx \quad \text{with} \quad \int_D p(x) \, dx = 1
  \]
- properties
  - $E[af(x)] = aE[f(x)]$
  - $E[\sum_i f(X_i)] = \sum_i E[f(X_i)]$ for independent random variables $X_i$
- example for uniform $p$
  \[
  E_p[\cos(x)] = \int_0^\pi \cos(x) \, \frac{1}{\pi} \, dx = \frac{1}{\pi} (-\sin \pi + \sin 0) = 0
  \]
motivation: quantifies the error of a Monte Carlo algorithm

variance $V$ of a function is the expected deviation of the function from its expected value $V[f(x)] = E[(f(x) - E[f(x)])^2]$

properties
- $V[af(x)] = a^2V[f(x)]$
- $V[f(x)] = E[(f(x))^2] - E[f(x)]^2$
- $\sum_i V[f(X_i)] = V[\sum_i f(X_i)]$ for independent random variables $X_i$
Monte Carlo Estimator

Uniform Random Variables

- motivation: approximation of the integral in the rendering equation
- goal: computation of $\int_a^b f(x) \, dx$
- uniformly distributed random variables $X_i \in [a, b]$
- probability density function $p(x) = \frac{1}{b-a}$ (constant and integration to one)
- Monte Carlo estimator $F_N = \frac{b-a}{N} \sum_{i=1}^{N} f(X_i)$
- expected value of $F_N$ is equal to the integral $\int_a^b f(x) \, dx$
  - $E[F_N] = \int_a^b f(x) \, dx$
- variance $V = \frac{1}{N-1} \sum_{i=1}^{N} [f(X_i) - E[F_N]]^2$
  - convergence rate of $O(\sqrt{N})$
  - independent from the dimensionality
  $\Rightarrow$ appropriate for high-dimensional integrals
Monte Carlo Estimator
Uniform Random Variables

\[ E[F_N] = E \left[ \frac{b-a}{N} \sum_{i=1}^{N} f(X_i) \right] \]
\[ = \frac{b-a}{N} \sum_{i=1}^{N} E[f(X_i)] \]
\[ = \frac{b-a}{N} \sum_{i=1}^{N} \int_{a}^{b} f(x)p(x)dx \]
\[ = \frac{b-a}{N} \sum_{i=1}^{N} \int_{a}^{b} f(x) \frac{1}{b-a} dx \]
\[ = \frac{1}{N} \sum_{i=1}^{N} \int_{a}^{b} f(x)dx \]
\[ = \int_{a}^{b} f(x)dx \]
Examples - Uniform Random Variables

- integral \( \int_0^1 5x^4 \, dx = 1 \)
- estimator \( F_N = \frac{1}{N} \sum_{i=1}^{N} 5X_i^4 \)
- for an increasing number of uniformly distributed random variables \( X_i \), the estimator converges to one

\[
F_N = \frac{b-a}{N} \sum_{i=1}^{N} f(X_i) \\
F_N = (b - a) \frac{1}{N} \sum_{i=1}^{N} f(X_i) \\
F_N = (b - a) \overline{f(x)} \\
E[F_N] = \int_{a}^{b} f(x) \, dx
\]

uniformly distributed random samples
Monte Carlo Estimator

Non-uniform Random Variables

Monte Carlo estimator \( F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)} \quad p(X_i) \neq 0 \)

\[ E[F_N] = E \left[ \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)} \right] \]

\[ = \frac{1}{N} \sum_{i=1}^{N} \int_{a}^{b} \frac{f(x)}{p(x)} p(x) \, dx \]

\[ = \frac{1}{N} \sum_{i=1}^{N} \int_{a}^{b} f(x) \, dx \]

\[ = \int_{a}^{b} f(x) \, dx \]
Monte Carlo Estimator

Multiple Dimensions

- samples $X_i$ are multidimensional
- e.g. $\int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} f(x, y, z) dx dy dz$
- uniformly distributed random samples
  $(x_0, y_0, z_0) \leq X_i = (x_i, y_i, z_i) \leq (x_1, y_1, z_1)$
- probability density function $p(X_i) = \frac{1}{x_1-x_0} \frac{1}{y_1-y_0} \frac{1}{z_1-z_0}$
- Monte Carlo estimator
  $F_N = \frac{(x_1-x_0)(y_1-y_0)(z_1-z_0)}{N} \sum_{i=1}^{N} f(X_i)$
- $N$ can be arbitrary, $N$ is independent from the dimensionality
Monte Carlo Estimator
Integration over a Hemisphere

- approximate computation of the irradiance at a point
  \[ E_i(p) = \int_{2\pi}^{0} L_i(p, \omega) \cos \theta d\omega \]
  \[ = \int_{0}^{2\pi} \int_{0}^{\frac{\pi}{2}} L_i(p, \theta, \phi) \cos \theta \sin \theta d\theta d\phi \]

- estimator
  \[ F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)} = \frac{1}{N} \sum_{i=1}^{N} \frac{L_i(p, \theta, \phi) \cos \theta \sin \theta}{p(\theta, \phi)} \]

- probability distribution
  - should be similar to the shape of the integrand
  - as incident radiance is weighted with \( \cos \theta \), it is appropriate to generate more samples close to the top of the hemisphere
  - \( p(\omega) \propto \cos \theta \)
Monte Carlo Estimator

Integration over a Hemispher

- probability distribution (cont.)
  \[ \int_{2\pi} \ c \ p(\omega) d\omega = 1 \]
  \[ \int_0^{2\pi} \int_0^{\frac{\pi}{2}} c \ \cos \theta \sin \theta \ d\theta \ d\phi = 1 \]
  \[ c \ \frac{2\pi}{1+1} = 1 \]
  \[ c = \frac{1}{\pi} \]
  \[ p(\theta, \phi) = \frac{\cos \theta \sin \theta}{\pi} \]

- estimator
  \[ F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{L_i(p, \theta, \phi) \cos \theta \sin \theta}{p(\theta, \phi)} \]
  \[ = \frac{\pi}{N} \sum_{i=1}^{N} L_i(p, \theta, \phi) \]
Monte Carlo Integration

Steps

- choose an appropriate probability density function
- generate random samples according to the PDF
- evaluate the function for all samples
- average the weighted function values
Monte Carlo Estimator

Error

- variance
  \[ V = \frac{1}{N} \int_a^b \left( \frac{f(x)}{p(x)} - F_N \right)^2 p(x) \, dx \]
- estimator
  \[ V_N = \frac{1}{N-1} \sum_{i=1}^N \left[ f(X_i) - F_N \right]^2 \]
- for increasing \( N \)
  - the variance decreases with \( O(N) \)
  - the standard deviation decreases with \( O(N^{\frac{1}{2}}) \)
- variance is perceived as noise
Monte Carlo Estimator

Variance Reduction / Error Reduction

- importance sampling
  - motivation: contributions of larger function values are more important
  - PDF should be similar to the shape of the function
  - optimal PDF \( p(x) = \frac{f(x)}{\int f(x)dx} \)
  - e.g., if incident radiance is weighted with \( \cos \theta \), the PDF should choose more samples close to the normal direction
- stratified sampling
  - domain subdivision into strata does not increase the variance
- multi-jittered sampling
  - alternative to random samples for, e.g., uniform sampling of area lights
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  - transforming between distributions
  - 2D sampling
  - examples


**Inversion Method**

- mapping of a uniform random variable to a goal distribution
- discrete example
  - four outcomes with probabilities \( p_1, p_2, p_3, p_4 \) and \( \sum_i p_i = 1 \)
- computation of the cumulative distribution function \( P(i) = \sum_{j=1}^{i} p_j \)
Inversion Method

- **discrete example cont.**
  - take a uniform random variable $\xi$
  - $P^{-1}(\xi)$ has the desired distribution

- **continuous case**
  - $P$ and $P^{-1}$ are continuous functions
  - start with the desired PDF $p(x)$
  - compute $P(x) = \int_0^x p(x') dx'$
  - compute the inverse $P^{-1}(x)$
  - obtain a uniformly distributed variable
  - compute $X_i = P^{-1}(\xi)$ which adheres to $p(x)$
Inversion Method

Example 1

- power distribution \( p(x) \propto x^n \)
  - e.g., for sampling the Blinn microfacet model
- computation of the PDF
  - \( \int_0^1 c x^n \, dx = 1 \Rightarrow c \left. \frac{x^{n+1}}{n+1} \right|_0^1 = 1 \Rightarrow c = n + 1 \)
- PDF \( p(x) = (n + 1)x^n \)
- CDF \( P(x) = \int_0^x p(x') \, dx' = x^{n+1} \)
- inverse of the CDF \( P^{-1}(x) = \sqrt[1+n]{x} \)
- sample generation
  - generate uniform random samples \( 0 \leq \xi \leq 1 \)
  - \( X = \sqrt[n+1]{\xi} \) are samples from the power distribution \( p(x) = (n + 1)x^n \)
**Inversion Method**

**Example 2**

- exponential distribution \( p(x) \propto e^{-ax} \)
  - e.g., for considering participating media

- computation of the PDF
  - \( \int_0^\infty c \ e^{-ax} \, dx = -\frac{c}{a} \ e^{-ax} \bigg|_0^\infty = \frac{c}{a} = 1 \)
  - PDF \( p(x) = a \ e^{-ax} \)
  - CDF \( P(x) = \int_0^x p(x') \, dx' = 1 - e^{-ax} \)
  - inverse of the CDF \( P^{-1}(x) = -\frac{\ln(1-x)}{a} \)

- sample generation
  - generate uniform random samples \( 0 \leq \xi \leq 1 \)
  - \( X = -\frac{\ln(1-\xi)}{a} \) are samples from the power distribution \( p(x) = a \ e^{-ax} \)
Inversion Method
Example 3

- piecewise-constant distribution
  - e.g., for environment lighting

\[
f(x) = \begin{cases} 
v_0 & x_0 \leq x < x_1 \\
v_1 & x_1 \leq x < x_2 \\
\vdots & \vdots \\
\end{cases}
\]

\[x_i = \Delta \cdot i\]

\[\Delta = \frac{1}{N}\]

- PDF \[p(x) = \frac{1}{c} f(x)\]
  - with \[c = \int_0^1 f(x) \, dx = \sum_{i=0}^{N-1} \Delta \cdot v_i = \frac{\sum_{i=0}^{N-1} v_i}{N}\]
Inversion Method
Example 3

- **CDF**

  \[ P(x_0) = 0 \]
  \[ P(x_1) = \int_{x_0}^{x_1} p(x) \, dx = \Delta \cdot \frac{v_0}{c} = \frac{v_0}{N_c} = P(x_0) + \frac{v_0}{N_c} \]
  \[ P(x_2) = \int_{x_0}^{x_2} p(x) \, dx = \int_{x_0}^{x_1} p(x) \, dx + \int_{x_1}^{x_2} p(x) \, dx = P(x_1) + \frac{v_1}{N_c} \]
  \[ P(x_i) = P(x_{i-1}) + \frac{v_{i-1}}{N_c} \]

- **CDF is linear between** \( x_i \) **and** \( x_{i+1} \) **with slope** \( \frac{v_i}{c} \)

- **sample generation**
  - generate uniform random samples \( 0 \leq \xi \leq 1 \)
  - compute \( x_i \) with \( P(x_i) \leq \xi \) and \( \xi < P(x_{i+1}) \)
  - compute \( d \) with \( P(x_i) + d(P(x_{i+1}) - P(x_i)) = \xi \)
  - \( X = x_i + d(x_{i+1} - x_i) = x_i + \frac{d}{N} \) are samples from \( p(x) = \frac{1}{c} f(x) \)
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  - 2D sampling
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Rejection Method

- draws samples according to a function $f(x)$
  - dart-throwing approach
  - works with a PDF $p(x)$ and a scalar $c$ with $f(x) < c \cdot p(x)$
- properties
  - $f(x)$ is not necessarily a PDF
  - PDF, CDF and inverse CDF do not have to be computed
  - simple to implement
  - useful for debugging purposes
- sample generation
  - generate a uniform random sample $0 \leq \xi < 1$
  - generate a sample $X$ according to $p(x)$
  - accept $X$ if $\xi \cdot c \cdot p(X) \leq f(X)$
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Transforming Between Distributions

- computation of a resulting PDF, when a function is applied to samples from an arbitrary distribution
  - random variables $X_i$ are drawn from $p_x(x)$
  - bijective transformation (one-to-one mapping) $Y_i = y(X_i)$
  - How does the distribution $p_y(y)$ look like?

$$Pr\{Y \leq y(x)\} = Pr\{X \leq x\}$$

$$p_y(y) = P_y(y(x)) = P_x(x)$$

$$p_y(y) = \frac{p_x(x)}{|y'(x)|}$$

- example $p_x(x) = 2x \quad 0 \leq x \leq 1$
  - $y(x) = \sin x \quad x(y) = \arcsin y$
  - $y'(x) = \cos x$
  - $p_y(y) = \frac{p_x(x)}{\cos x} = \frac{2 \arcsin y}{\sqrt{1-y^2}}$
Transforming Between Distributions

- multiple dimensions
  - $X_i$ is an n-dimensional random variable
  - $Y_i = T(X_i)$ is a bijective transformation

- transformation of the PDF
  
  $$ p_y(y) = \frac{p_x(x)}{|J_T(x)|} \quad J_T(x) = \begin{pmatrix} \frac{\partial T_1}{\partial x_1} & \ldots & \frac{\partial T_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial T_n}{\partial x_1} & \ldots & \frac{\partial T_n}{\partial x_n} \end{pmatrix} $$

- example (polar coordinates)
  - samples $(r, \theta)$ with density $p(r, \theta)$
  - corresponding density $p(x, y)$ with $x = r \cos \theta$ and $y = r \sin \theta$
  - $J_T(x) = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}$
  - $|J_T(x)| = r(\cos^2 \theta + \sin^2 \theta) = r$
  - $p(x, y) = \frac{1}{r} p(r, \theta)$
  - $p(r, \theta) = r \cdot p(x, y)$
Transforming Between Distributions

- example (spherical coordinates)
  - $x = r \sin \theta \cos \phi$
  - $y = r \sin \theta \sin \phi$
  - $z = r \cos \theta$
  - $p(r, \theta, \phi) = r^2 \sin \theta \cdot p(x, y, z)$

- example (solid angle)
  - $Pr\{\omega \in \Omega\} = \int_{\Omega} p(\omega) d\omega$
  - $d\omega = \sin \theta \ d\theta \ d\phi$
  - $p(\theta, \phi) = \sin \theta \cdot p(\omega)$
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Concept

- generation of samples from a 2D joint density function $p(x, y)$
- general case
  - compute the marginal density function $p_x(x) = \int p(x, y) dy$
  - compute the conditional density function $p_y(y|x) = \frac{p(x, y)}{p_x(x)}$
  - generate a sample $X$ according to $p_x(x)$
  - generate a sample $Y$ according to $p_y(y|X) = \frac{p(x, y)}{p_x(X)}$
- marginal density function
  - integral of $p(x, y)$ for a particular $x$ over all $y$-values
- conditional density function
  - density function for $y$ given a particular $x$
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Uniform Sampling of a Hemisphere

- PDF is constant with respect to a solid angle \( p(\omega) = c \)
  \[
  \int_{2\pi} p(\omega) \, d\omega = 1 \Rightarrow c \int_{2\pi} d\omega = 1 \Rightarrow c = \frac{1}{2\pi}
  \]
- \( p(\omega) = \frac{1}{2\pi} \Rightarrow p(\theta, \phi) = \frac{\sin \theta}{2\pi} \)
- marginal density function
  \[
  p_\theta(\theta) = \int_0^{2\pi} p(\theta, \phi) \, d\phi = \int_0^{2\pi} \frac{\sin \theta}{2\pi} \, d\phi = \sin \theta
  \]
- conditional density for \( \phi \)
  \[
  p_\phi(\phi|\theta) = \frac{p(\theta, \phi)}{p_\theta(\theta)} = \frac{1}{2\pi}
  \]
- inversion method
  \[
  P_\theta(\theta) = \int_0^\theta \sin \theta' \, d\theta' = -\cos \theta + 1
  \]
  \[
  P_\phi(\phi|\theta) = \int_0^\phi \frac{1}{2\pi} \, d\phi' = \frac{\phi}{2\pi}
  \]
Uniform Sampling of a Hemisphere

- inversion method cont.
  - inverse functions of the cumulative distribution functions
    - \( \theta = \arccos(1 - \xi_1) \)
    - \( \phi = 2\pi \xi_2 \)
  - generating uniformly sampled random values \( \xi_1 \) and \( \xi_2 \)
  - applying the inverse CDFs to obtain \( \theta \) and \( \phi \)
- conversion to Cartesian space
  - \( x = \sin \theta \cos \phi = \cos(2\pi \xi_2) \sqrt{1 - (1 - \xi_1)^2} \)
  - \( y = \sin \theta \sin \phi = \sin(2\pi \xi_2) \sqrt{1 - (1 - \xi_1)^2} \)
  - \( z = \cos \theta = 1 - \xi_1 \)
- \((x, y, z)^T\) is a normalized direction
Uniform Sampling of a Hemisphere

- illustration for \( \theta \)

\[ \theta = \arccos(1 - \xi_1) \]
Uniform Sampling of a Unit Disk

- PDF is constant with respect to area $p(x, y) = \frac{1}{\pi}$
- $p(r, \theta) = r \cdot p(x, y) \Rightarrow \frac{r}{\pi}$
- marginal density function
  - $p_r(r) = \int_0^{2\pi} p(r, \theta) d\theta = 2r$
- conditional density
  - $p_\theta(\theta|r) = \frac{p(r, \theta)}{p_r(r)} = \frac{1}{2\pi}$
- inversion method
  - $P_r(r) = \int_0^r 2r' dr' = r^2$
  - $P_\theta(\theta|r) = \int_0^{2\pi} \frac{1}{2\pi} d\theta' = \frac{\theta}{2\pi}$
Uniform Sampling of a Unit Disk

- inversion method cont.
  - inverse functions of the cumulative distribution functions
  - \( r = \sqrt{\xi_1} \)
  - \( \theta = 2\pi \xi_2 \)
  - generating uniformly sampled random values \( \xi_1 \) and \( \xi_2 \)
  - applying the inverse CDFs to obtain \( r \) and \( \theta \)

![Graph showing the relationship between \( \xi_1 \) and \( r \). The graph indicates that for smaller values of \( \xi_1 \), the value of \( r \) is generated with less samples.](image-url)
Uniform Sampling
of a Cosine-Weighted Hemisphere

- PDF is proportional to $\cos \theta$. 
  $p(\omega) \propto \cos \theta$
  \[ \int_{2\pi}^{2\pi+} c \ p(\omega) \ d\omega = 1 = \int_{0}^{2\pi} \int_{0}^{\pi} c \ \cos \theta \sin \theta \ d\theta \ d\phi = c \ 2\pi \int_{0}^{\pi/2} \cos \theta \sin \theta \ d\theta = c \ 2\pi \frac{1}{2} = 1 \]

- marginal density function
  \[ p(\theta, \phi) = \frac{1}{\pi} \cos \theta \sin \theta \]

- conditional density for $\phi$
  \[ p(\phi|\theta) = \frac{p(\theta,\phi)}{p(\theta)} = \frac{1}{2\pi} \]

- inversion method
  \[ P(\theta) = \int_{0}^{\theta} 2 \cos \theta' \sin \theta' \ d\theta' = 2 \left[ -\frac{\cos^2 \theta'}{2} \right]_{0}^{\theta} = 2 \left( -\frac{\cos^2 \theta}{2} + \frac{1}{2} \right) = \sin^2 \theta \]
  \[ P(\phi|\theta) = \int_{0}^{\phi} \frac{1}{2\pi} \ d\theta' = \frac{\phi}{2\pi} \]
Uniform Sampling of a Cosine-Weighted Hemisphere

- inversion method cont.
  - inverse functions of the cumulative distribution functions
    - \( \theta = \arcsin(\sqrt{\xi_1}) \)
    - \( \phi = 2\pi \xi_2 \)
  - generating uniformly sampled random values \( \xi_1 \) and \( \xi_2 \)
  - applying the inverse CDFs to obtain \( \theta \) and \( \phi \)

- conversion to Cartesian space
  - \( x = \sin \theta \cos \phi = \cos(2\pi \xi_2) \sqrt{\xi_1} \)
  - \( y = \sin \theta \sin \phi = \sin(2\pi \xi_2) \sqrt{\xi_1} \)
  - \( z = \cos \theta = \sqrt{1 - \xi_1} \)

- \( (x, y, z)^T \) is a normalized direction

- x- y- values uniformly sample a unit disk, i.e., cosine-weighted samples of the hemisphere can also be obtained by uniformly sampling a unit sphere and projecting the samples onto the hemisphere
Uniform Sampling of a Cosine-Weighted Hemisphere

Illustration for $\theta$

$$\theta = \arcsin(\xi_1)$$

- Cosine-weighted hemisphere (top view, side view)
- Uniform hemisphere (top view)

generate less samples for smaller and larger angles $\theta$
Uniform Sampling of a Triangle

- sampling an isosceles right triangle of area 0.5
  - $u, v$ can be interpreted as Barycentric coordinates
  - can be used to generate samples for arbitrary triangles
- $p(u, v) = 2$
- marginal density function
  - $p_u(u) = \int_0^{1-u} p(u, v) \, dv = 2 \int_0^{1-u} dv = 2(1 - u)$
- conditional density
  - $p_v(v|u) = \frac{p(u,v)}{p_u(u)} = \frac{1}{1-u}$
- inversion method
  - $P_u(u) = \int_0^u 2 - 2u' \, du' = 2u - u^2$
  - $P_v(v|u) = \int_0^v \frac{1}{1-u} \, dv' = \frac{v}{1-u}$
Uniform Sampling of a Triangle

- inversion method cont.
  - inverse functions of the cumulative distribution functions
  - \( u = 1 - \sqrt{\xi_1} \)
  - \( v = \xi_2 \sqrt{\xi_1} \)
  - generating uniformly sampled random values \( \xi_1 \) and \( \xi_2 \)
  - applying the inverse CDFs to obtain \( u \) and \( v \)
Piecwise-Constant 2D Distribution

- \( n_u \times n_v \) samples defined over \((u, v) \in [0, 1]^2\)
  - e.g., an environment map
- \( f(u, v) \) is defined by a set of \( n_u \times n_v \) values \( f[u_i, v_i] \)
  - \( u_i \in [0, \ldots, n_u - 1] \quad v_i \in [0, \ldots, n_v - 1] \)
  - \( f[u_i, v_i] \) is the value of \( f(u, v) \) in the range \( \left[ \frac{i}{n_u}, \frac{i+1}{n_u} \right] \times \left[ \frac{j}{n_v}, \frac{j+1}{n_v} \right] \)
  - \( f(u, v) = f[u_i, v_i] \) with \( \tilde{u} = \lfloor n_u u \rfloor \) and \( \tilde{v} = \lfloor n_v v \rfloor \)
- integral over the domain
  - \( I_f = \int \int f(u, v) \, du \, dv = \frac{1}{n_u n_v} \sum_i \sum_j f[u_i, v_j] \)
- PDF
  - \( p(u, v) = \frac{1}{I_f} f(u, v) = \frac{1}{I_f} f[\tilde{u}, \tilde{v}] \)
Piecewise-Constant 2D Distribution

- marginal density function
  $$p_v(v) = \int p(u,v) \, du = \frac{1}{I_f} \frac{1}{n_v} \sum_i f[u_i, \tilde{v}]$$
  - piecewise-constant 1D function
  - defined by $n_v$ values $p_v[\tilde{v}]$

- conditional density
  $$p_u(u|v) = \frac{p(u,v)}{p_v(v)} = \frac{1}{I_f} \frac{f[\tilde{u}, \tilde{v}]}{p[\tilde{v}]}$$
  - piecewise-constant 1D function

- sample generation
  - see example 3 of the inversion method
**Piecewise-Constant 2D Distribution**

- environment map

- low-resolution of the marginal density function and the conditional distributions for rows
  - first, a row is selected according to the marginal density function
  - then, a column is selected from the row's 1D conditional distribution

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