Advanced Computer Graphics
Summary

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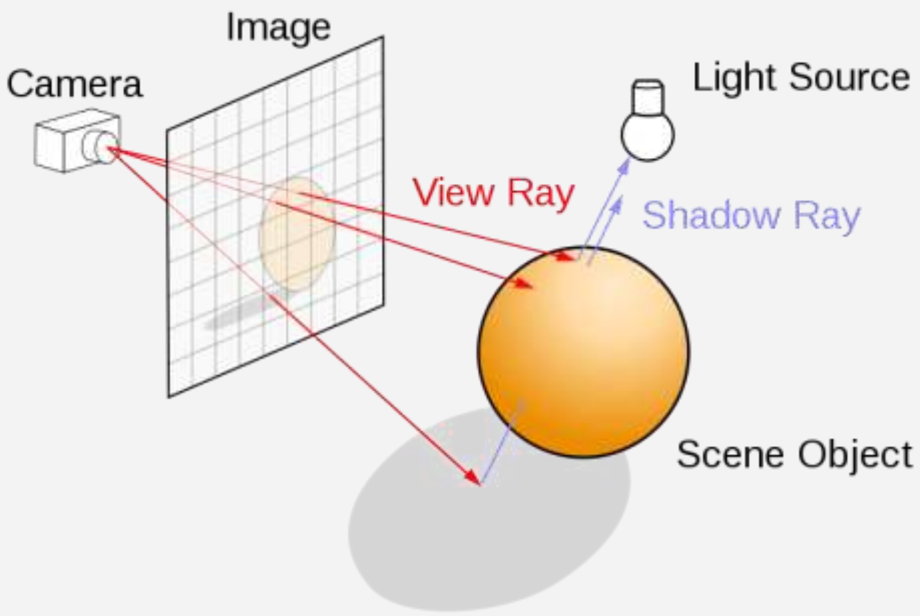
Outline

- Introduction
- Ray-object intersections
- Light
- Materials
- Radiosity
- Stochastic Raytracing

Ray Tracing

- Tracing rays through a scene to compute the radiance that is perceived by a sensor, i.e. transported along rays
- Tracing a path from a camera through a pixel position of a virtual image plane to compute the color / radiance of an object that is visible along the path



[Wikipedia: Ray Tracing]

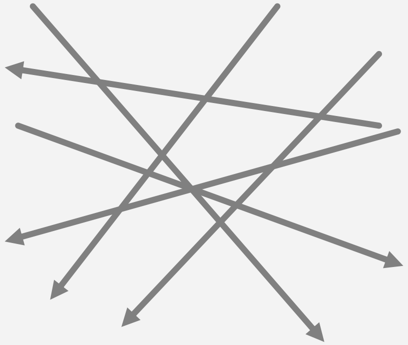
Light

- Is modeled as geometric rays
  - Travels in straight lines (e.g., no diffraction / bending)
  - Travels at infinite speed (steady state of light is computed)
  - Is emitted by light sources
  - Is absorbed / scattered at surfaces

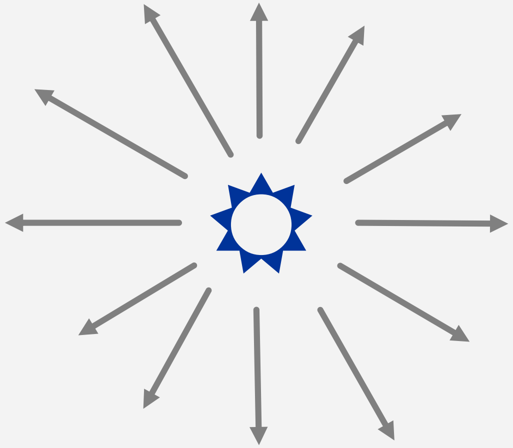
Measuring Light

- Radiance
  - Characterizes strength and direction of radiation / light
  - Is measured by sensors
  - Is computed in computer-generated images
  - Is preserved along lines in space
  - Does not change with distance

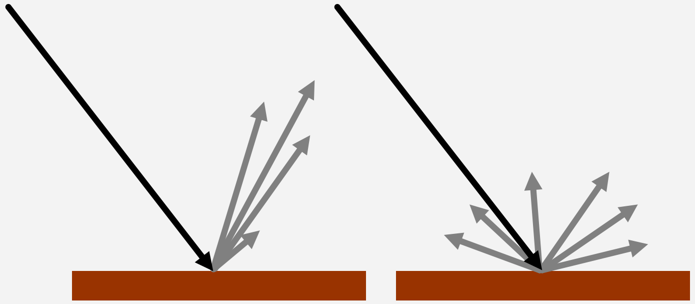
Aspects



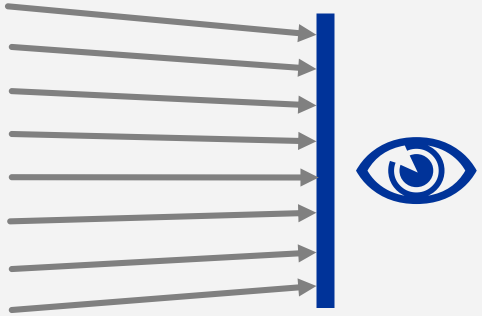
Light / radiance
travels along rays



Light / radiance
is emitted at
light sources



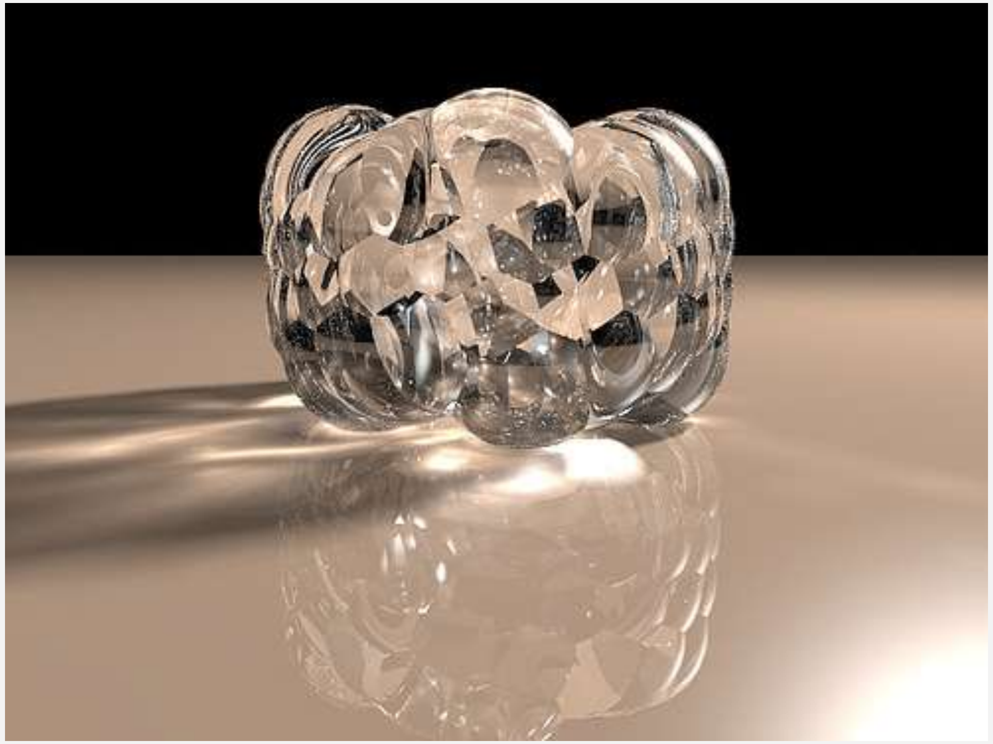
Incoming light /
radiance is absorbed
and scattered at surfaces



Cameras capture
light / radiance

Ray Tracing - Capabilities

- Reflection
- Refraction
- Soft shadows
- Caustics
- Diffuse interreflections
- Specular interreflections
- Depth of field
- Motion blur



[sean.seanie, www.flickr.com]
rendered with POVray 3.7

Ray Tracing - Challenges

- Ray shooting (ray-object intersections)
- Number of rays (quality vs. costs)
  - Approximately solving the Rendering equation
- Recursion depth (quality vs. costs)

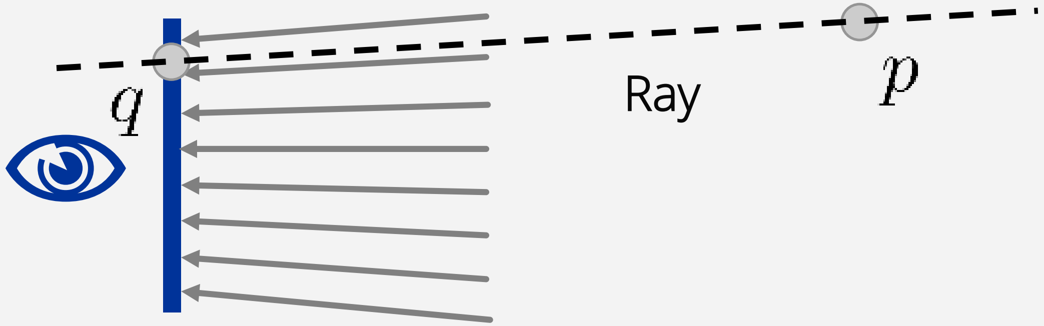
Ray Tracing vs. Rasterization

- Rasterization
  - Given a set of viewing rays and a primitive, efficiently compute the subset of rays hitting the primitive
  - Loop over all primitives
  - Implicit ray representation
- Ray tracing
  - Given a viewing ray and a set of primitives, efficiently compute the subset of primitives hit by the ray
  - Loop over all viewing rays
  - Explicit ray representation

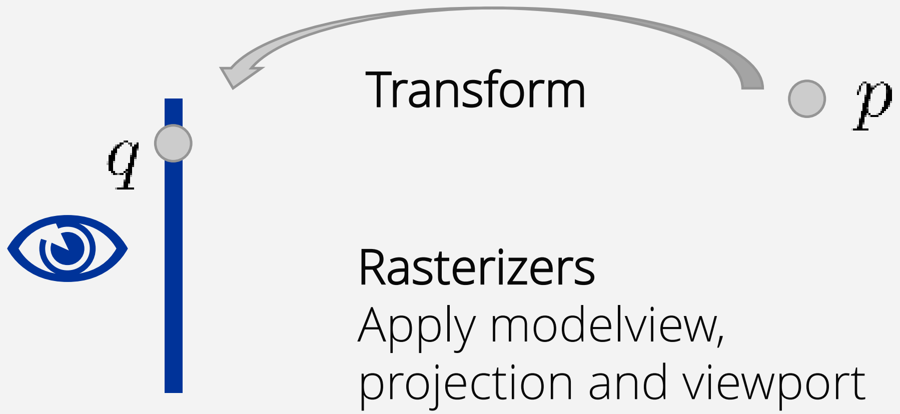
[Ray Tracing Course: SIGGRAPH 2005]

Ray Tracing vs. Rasterization

– Solve the same problem



Ray Tracers
Compute ray-object intersections to estimate q from p



Rasterizers
Apply modelview, projection and viewport transform to p in order to estimate q

[Ray Tracing Course: SIGGRAPH 2005]

Ray Tracing vs. Rasterization

- Rasterization
  - Well-established, parallelizable algorithms
  - Popular in interactive applications
  - Specialized realizations of global illumination effects
- Ray tracing
  - Natural incorporation of numerous visual effects
  - Unified algorithms for global illumination effects
  - Trade-off between quality and performance

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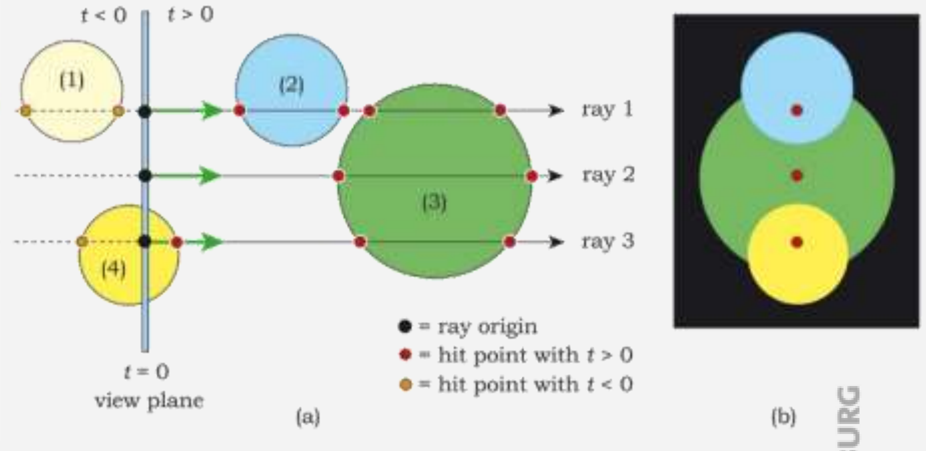
Motivation

- Rays
  - A half-line specified by an origin / position **o** and a direction **d**
  - Parametric form $\mathbf{r}(t) = \mathbf{o} + t\mathbf{d}$ with $0 \leq t \leq \infty$
- Nearest intersection with all objects has to be computed, i.e. intersection with minimal $t \geq 0$
- In implementations, usually $t \geq \varepsilon$ to avoid self-intersections, e.g., if rays start at object surfaces



[Suffern]

Implicit Surfaces

- Implicit functions implicitly define a set of surface points
- For a surface point (x,y,z), an implicit function *f*(x,y,z) is zero
- An intersection occurs, if a point on a ray satisfies the implicit equation $f(x,y,z) = f(\mathbf{r}(t)) = f(\mathbf{o} + t\mathbf{d}) = 0$
- E.g., all points **p** on a plane with surface normal **n** and offset **r** satisfy the equation $\mathbf{n} \cdot (\mathbf{p} - \mathbf{r}) = 0$
- The intersection with a ray can be computed based on *t*

$\mathbf{n} \cdot (\mathbf{o} + t\mathbf{d} - \mathbf{r}) = 0 \quad t = \frac{(\mathbf{r}-\mathbf{o})\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{d}}$ if **d** is not orthogonal to **n**

Implicit Surfaces - Normal

- Perpendicular to the surface
- Given by the gradient of the implicit function

$$\mathbf{n} = \nabla f(\mathbf{p}) = \left(\frac{\partial f(\mathbf{p})}{\partial x}, \frac{\partial f(\mathbf{p})}{\partial y}, \frac{\partial f(\mathbf{p})}{\partial z}\right)$$

- E.g., for a point p on a plane $f(\mathbf{p}) = \mathbf{n} \cdot (\mathbf{p} - \mathbf{r}) = 0$

$$\mathbf{n} = \nabla f(\mathbf{p}) = (n_x, n_y, n_z)$$

Triangle

– Parametric representation (barycentric coords)

$$\mathbf{p}(b_1, b_2) = (1 - b_1 - b_2)\mathbf{p_0} + b_1\mathbf{p_1} + b_2\mathbf{p_2}$$

$$b_1 \geq 0 \qquad b_2 \geq 0 \qquad b_1 + b_2 \leq 1$$

– Intersection is computed using a linear system

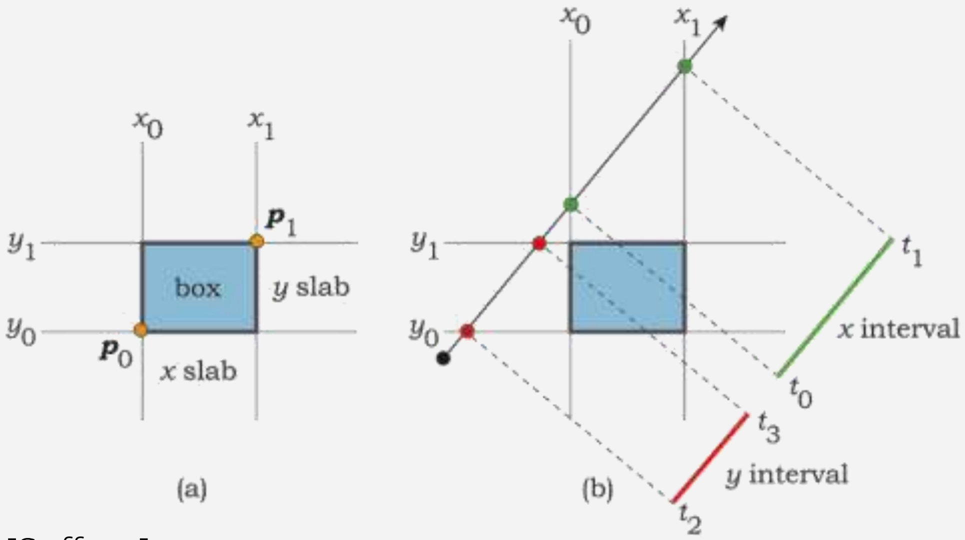
$$\mathbf{o} + t\mathbf{d} = (1 - b_1 - b_2)\mathbf{p_0} + b_1\mathbf{p_1} + b_2\mathbf{p_2}$$

– Solution (non-degenerated triangles, not parallel to ray)

$$\begin{pmatrix} t \\ b_1 \\ b_2 \end{pmatrix} = \frac{1}{(\mathbf{d} \times \mathbf{e_2}) \cdot \mathbf{e_1}} \begin{pmatrix} (\mathbf{s} \times \mathbf{e_1}) \cdot \mathbf{e_2} \\ (\mathbf{d} \times \mathbf{e_2}) \cdot \mathbf{s} \\ (\mathbf{s} \times \mathbf{e_1}) \cdot \mathbf{d} \end{pmatrix} \qquad \begin{aligned} \mathbf{e_1} &= \mathbf{p_1} - \mathbf{p_0} \\ \mathbf{e_2} &= \mathbf{p_2} - \mathbf{p_0} \\ \mathbf{s} &= \mathbf{o} - \mathbf{p_0} \end{aligned}$$

Axis-Aligned (Bounding) Box AABB

- Boxes are represented by slabs
- Intersections of rays with slabs are analyzed to check for ray-box intersection
  - E.g. non-overlapping ray intervals within different slabs indicate that the ray misses the box



[Suffern]

$$\mathbf{n} \cdot (\mathbf{o} + t\mathbf{d} - \mathbf{r}) = 0$$

$$t = \frac{(\mathbf{r}-\mathbf{o})\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{d}}$$

general case

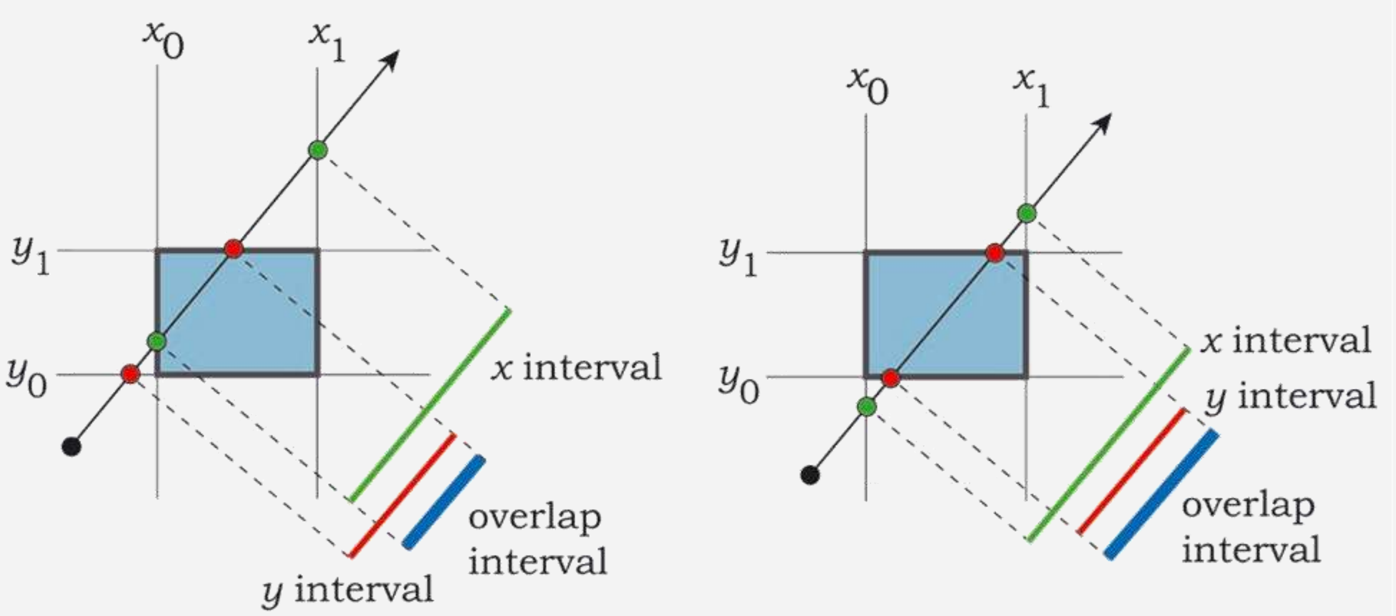
$$(1,0,0)^T \cdot (\mathbf{o} + t\mathbf{d} - (x_{0,1},0,0)^T) = 0$$

$$t_{x_{\min}, x_{\max}} = \frac{(x_{0,1} - o_x)}{d_x}$$

intersection with x-slab

Axis-Aligned (Bounding) Box AABB

– Overlapping ray intervals indicate intersections, e.g. $t_{xmin} < t_{ymax} \wedge t_{xmax} > t_{ymin} \Rightarrow$ intersection
(largest entering value *t* is smaller than the smallest leaving value *t*, only positive values t are considered)



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The Importance of Light Modeling

Light

↓

Color

↓

Material

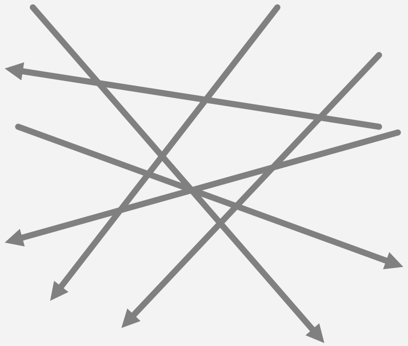
↓

Rendering equation

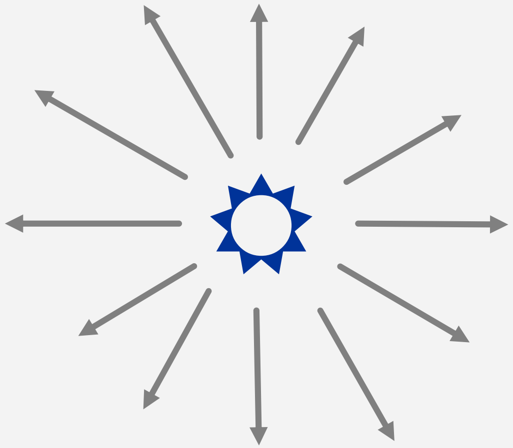
↓

Solving the rendering equation
(Computing pixel colors)

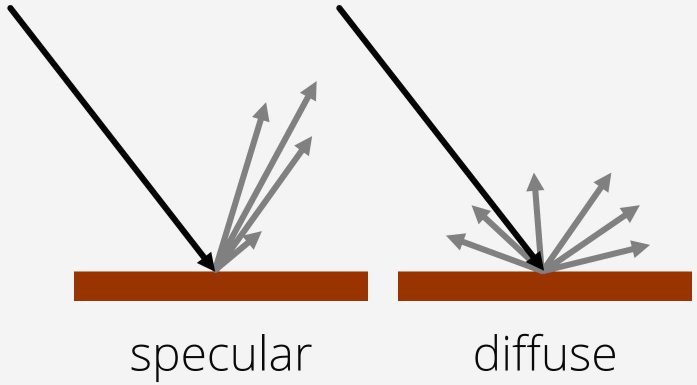
Light



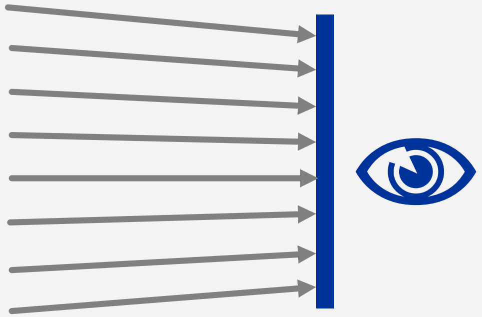
Light travels
along rays



Light is
emitted at
light sources



Incoming light /
is absorbed and
scattered at surfaces



Cameras
capture light

- How to quantify light/color? ⇨ Flux, Irradiance, Radiance
- How to quantify surface illumination? ⇨ Irradiance
- How to quantify pixel colors? ⇨ Radiance

Flux

- Radiant flux Φ
  - Power
  - Radiant energy, i.e. number of photons, per time
  - Brightness, e.g., number of photons emitted by a source per time

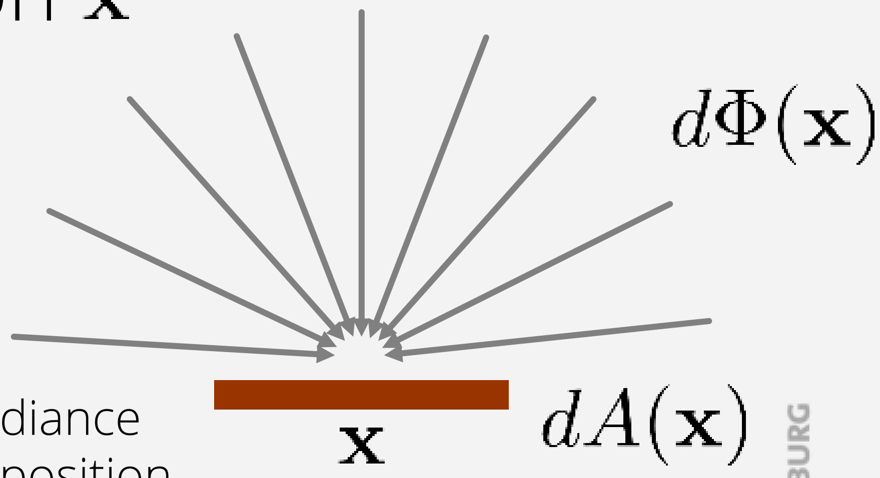
$$\Phi = \frac{dQ}{dt}$$

Flux is actually radiant energy per time.

As photons carry varying energy depending on their wavelength, number of photons per time is an approximation that improves the intuition behind flux.

Flux Density

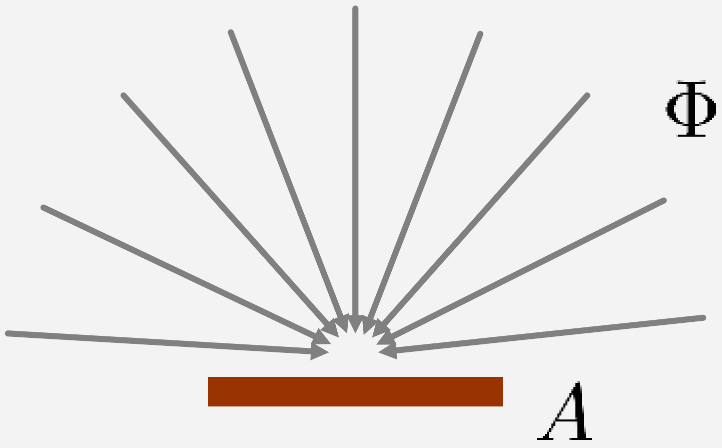
- Irradiance at a position $E(\mathbf{x})$?
  - Issues: position with zero area, no flux per position
  - Solution: infinitesimals, differentials, small quantities
- Consider a small amount of flux $d\Phi(\mathbf{x})$ incident to a small area $dA(\mathbf{x})$ around position $\mathbf{x}$
- For $dA(\mathbf{x}) \to 0$, we have $d\Phi(\mathbf{x}) \to 0$, and the ratio converges to the irradiance at $\mathbf{x}$: $E(\mathbf{x}) = \frac{d\Phi(\mathbf{x})}{dA(\mathbf{x})}$



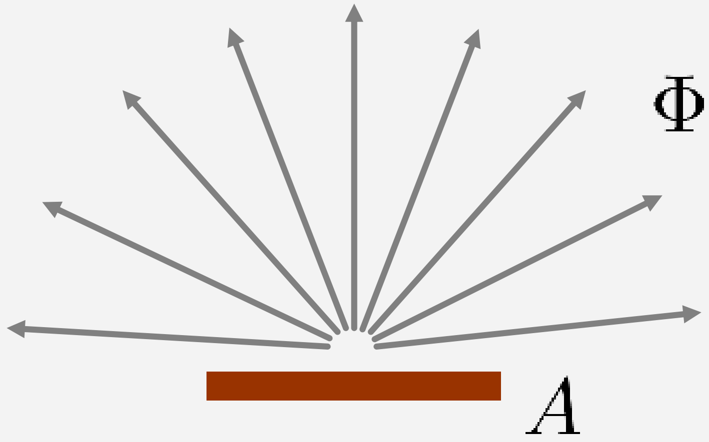
Irradiance at a position

Flux Density - Variants

– Irradiance E - incident / incoming flux per surface
– Radiosity B - outgoing flux (reflected plus emitted) per surface



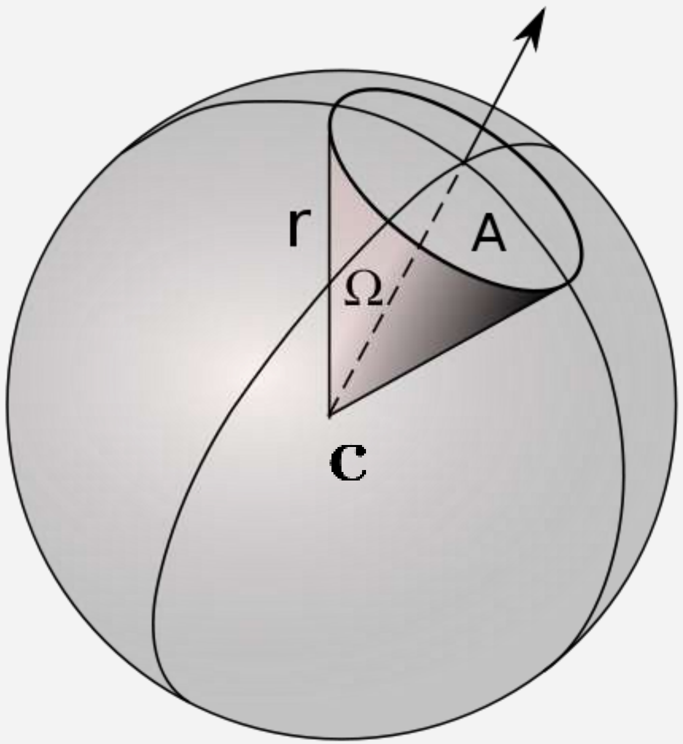
Irradiance – Incident flux per area



Radiosity – Outgoing flux per area

Solid Angle

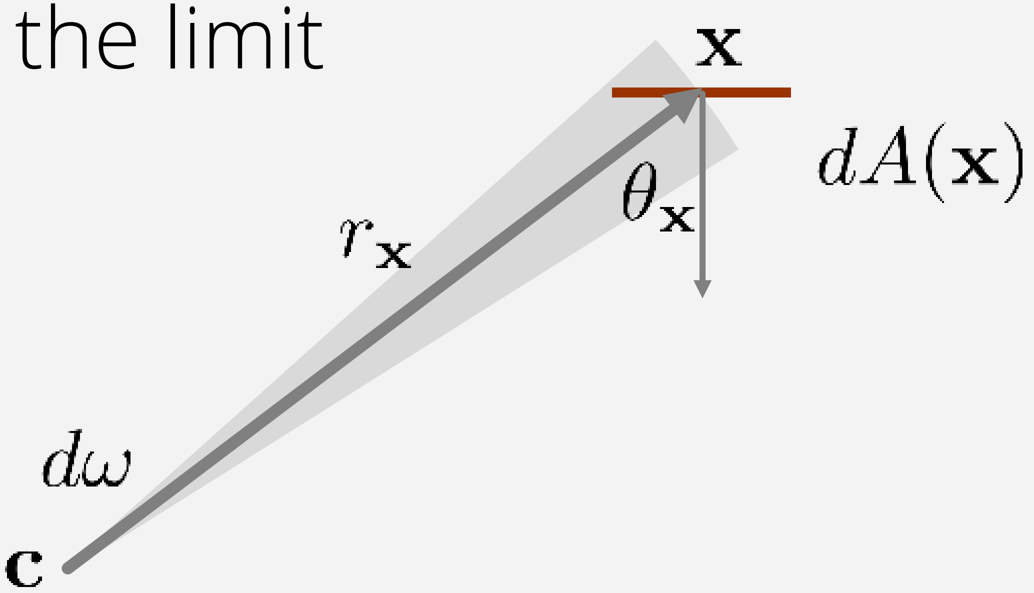
- Area of a sphere surface divided by the squared sphere radius $\Omega = \frac{A}{r^2}$
- E.g., solid angle of the entire sphere surface $\Omega = \frac{4\pi r^2}{r^2} = 4\pi$
  - Independent from the radius
- E.g., solid angle of a hemisphere $\Omega = \frac{1}{2}\frac{4\pi r^2}{r^2} = 2\pi$



Wikipedia: Raumwinkel

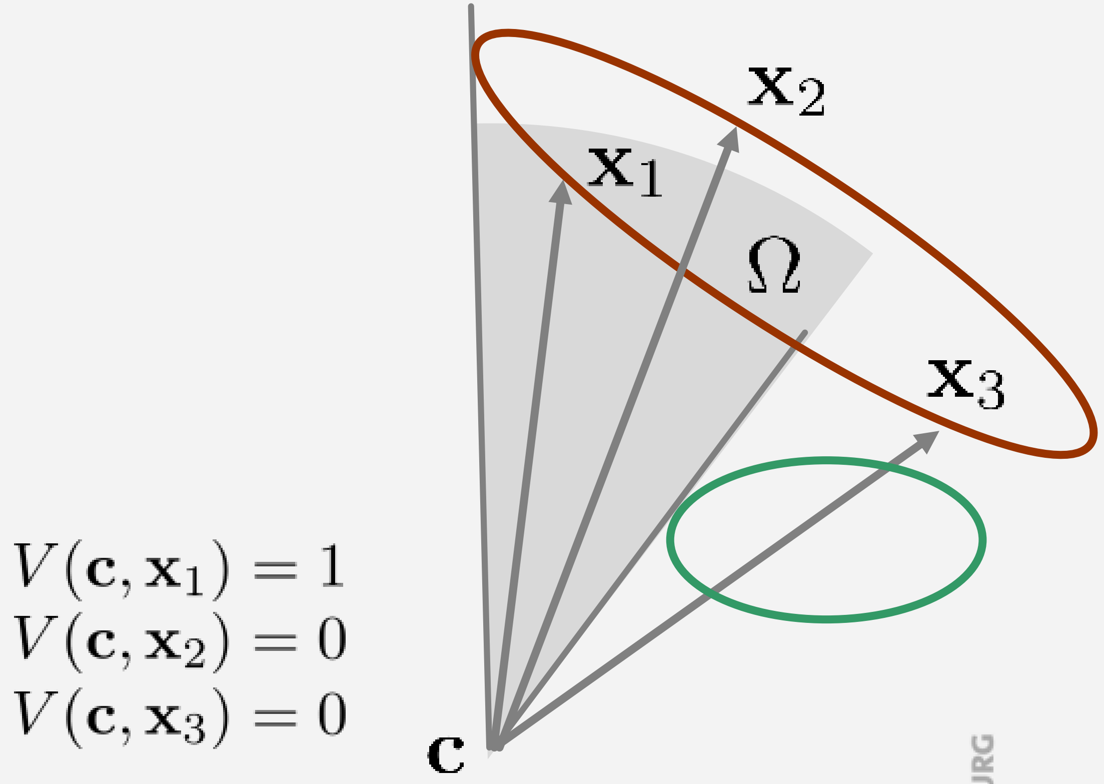
Infinitesimal Solid Angle and Surface Area

- $\Omega \approx \frac{A \cos \theta}{r^2}$ is an approximation
- If an infinitesimally small area $dA(\mathbf{x})$ at position $\mathbf{x}$ converges to zero, then the solid angle $d\omega$ also converges to zero and the relation $d\omega = \frac{dA(\mathbf{x}) \cos \theta_{\mathbf{x}}}{r_{\mathbf{x}}^2}$ is correct in the limit



Visibility Function

- Position $\mathbf{x}$ only contributes to $\int_{\text{Surface}} \frac{\cos\theta_{\mathbf{x}}}{r_{\mathbf{x}}^2} dA(\mathbf{x})$, if it is visible from $\mathbf{c}$
- Therefore,
  $\Omega = \int_{Surface} V(\mathbf{c},\mathbf{x}) \frac{\cos\theta_{\mathbf{x}}}{r_{\mathbf{x}}^2} dA(\mathbf{x})$
  with $V(\mathbf{c},\mathbf{x}) = 1$, if $\mathbf{x}$ is visible from $\mathbf{c}$ and $V(\mathbf{c},\mathbf{x}) = 0$, if $\mathbf{x}$ is not visible from $\mathbf{c}$

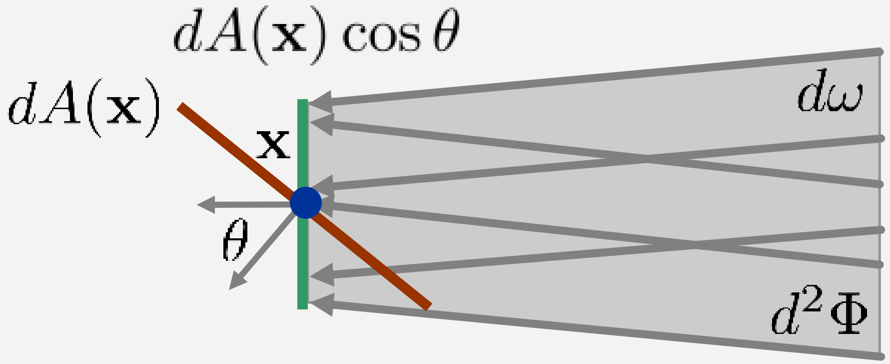


Radiance at a Position in a Direction

- Actual setting

  $$L(\mathbf{x}, \omega) = \frac{d^2\Phi}{dA(\mathbf{x}) \cdot \cos\theta \cdot d\omega}$$

  - Flux that is transported through an infinitesimally small cone



- Simplified notion

  $$L(\mathbf{x}, \omega)$$

  - Radiance L at position $\mathbf{x}$ in direction ω
  - Flux that is transported along a ray

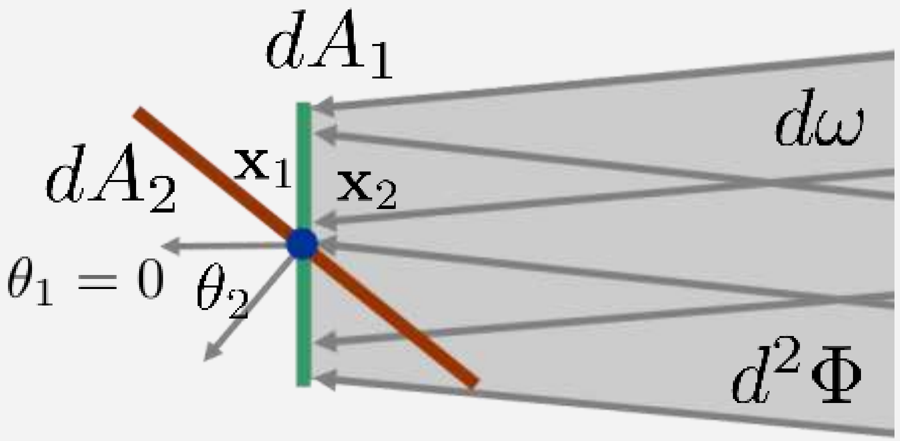
Radiance and Oriented Surfaces

– Two areas dA_1, dA_2 around positions $\mathbf{x}_1, \mathbf{x}_2$ with $\mathbf{x}_1 = \mathbf{x}_2$

– Angles between surface normal and flux direction ω:
  $\theta_1 = 0, \theta_2 \neq 0$

– Radiance at $\mathbf{x}_1$:
  $$L(\mathbf{x}_1, \omega) = \frac{d^2\Phi}{dA_1 \cdot \cos\theta_1 \cdot d\omega} = \frac{d^2\Phi}{dA^\perp \cdot d\omega}$$

– Radiance at $\mathbf{x}_2$:
  $$L(\mathbf{x}_2, \omega) = \frac{d^2\Phi}{dA_2 \cdot \cos\theta_2 \cdot d\omega} = \frac{d^2\Phi}{dA^\perp \cdot d\omega}$$



Radiance describes the flux within the grey area independent from the plane (sensor) orientation.

Irradiance and Oriented Surfaces

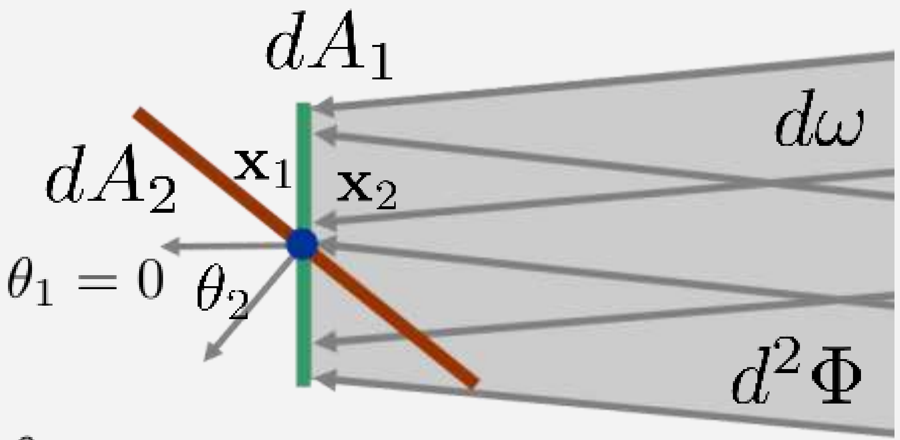
- Irradiance at $\mathbf{x}_1$: $E(\mathbf{x}_1) = \frac{d\Phi}{dA_1}$
- Irradiance at $\mathbf{x}_2$: $E(\mathbf{x}_2) = \frac{d\Phi}{dA_2}$
- $dA_2 = \frac{dA_1}{\cos\theta_2} \Rightarrow E(\mathbf{x}_2) = \cos\theta_2 \cdot E(\mathbf{x}_1)$

  $dA^{\perp} = dA_1 \cos\theta_1 = dA_2 \cos\theta_2 = dA_i \cos\theta_i$

$$\Rightarrow E^{\perp} = \frac{E(\mathbf{x}_1)}{\cos\theta_1} = \frac{E(\mathbf{x}_2)}{\cos\theta_2} = \frac{E(\mathbf{x}_i)}{\cos\theta_i}$$

i denotes an arbitrary orientation.

- Lambert's Cosine Law
  - Irradiance on a surface is proportional to the cosine of the angle between surface normal and flux direction



Irradiance describes the effect of the flux within the grey area onto a surface. I.e., the orientation of the surface with respect to the flux direction matters.

Conservation of Radiance

- Radiosity at $\mathbf{x}_1$: $B(\mathbf{x}_1) = \frac{d\Phi}{dA_1}$
- Irradiance at $\mathbf{x}_2$: $E(\mathbf{x}_2) = \frac{d\Phi}{dA_2} \neq B(\mathbf{x}_1)$
- Radiance at $\mathbf{x}_1$:

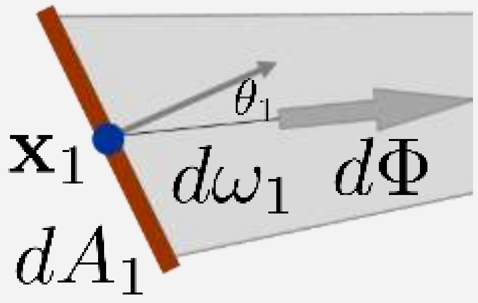
$$L(\mathbf{x}_1, \omega_1) = \frac{d^2\Phi}{dA_1 \cdot \cos\theta_1 \cdot d\omega_1} \quad d\omega_1 = \frac{dA_2 \cdot \cos\theta_2}{r^2}$$

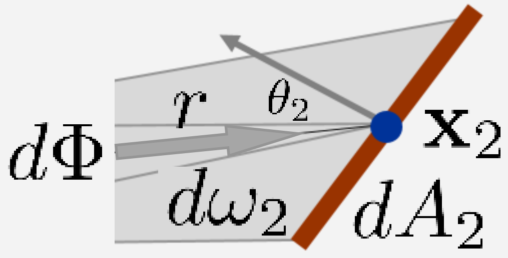
$$L(\mathbf{x}_1, \omega_1) = \frac{r^2 \cdot d^2\Phi}{dA_1 \cdot \cos\theta_1 \cdot dA_2 \cdot \cos\theta_2}$$

- Radiance at $\mathbf{x}_2$:

$$L(\mathbf{x}_2, \omega_2) = \frac{d^2\Phi}{dA_2 \cdot \cos\theta_2 \cdot d\omega_2} \quad d\omega_2 = \frac{dA_1 \cdot \cos\theta_1}{r^2}$$

$$L(\mathbf{x}_2, \omega_2) = \frac{r^2 \cdot d^2\Phi}{dA_1 \cdot \cos\theta_1 \cdot dA_2 \cdot \cos\theta_2} = L(\mathbf{x}_1, \omega_1)$$





Conservation of radiance. Radiance describes flux transported along a ray.

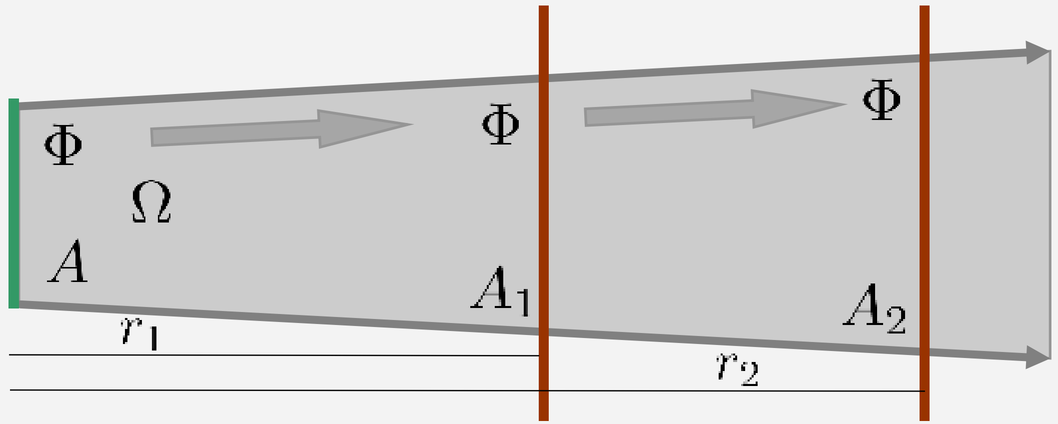
UNI FREIBURG

Discussion – Inverse Square Law

- Irradiance at an illuminated surface decreases quadratically with the distance from a light source
  - Surfaces appear darker with growing distance from light
  - Flux generated at A, arriving at A_1 and A_2: $L \cdot A \cdot \Omega$
  - Areas

    $A_1 \sim \Omega \cdot r_1^2 \quad A_2 \sim \Omega \cdot r_2^2$
  - Irradiances

    $E_1 \sim \frac{\Phi}{A_1} = \frac{L \cdot A \cdot \Omega}{\Omega \cdot r_1^2} \quad E_2 \sim \frac{\Phi}{A_2} = \frac{L \cdot A \cdot \Omega}{\Omega \cdot r_2^2} \quad E \sim \frac{1}{r^2}$

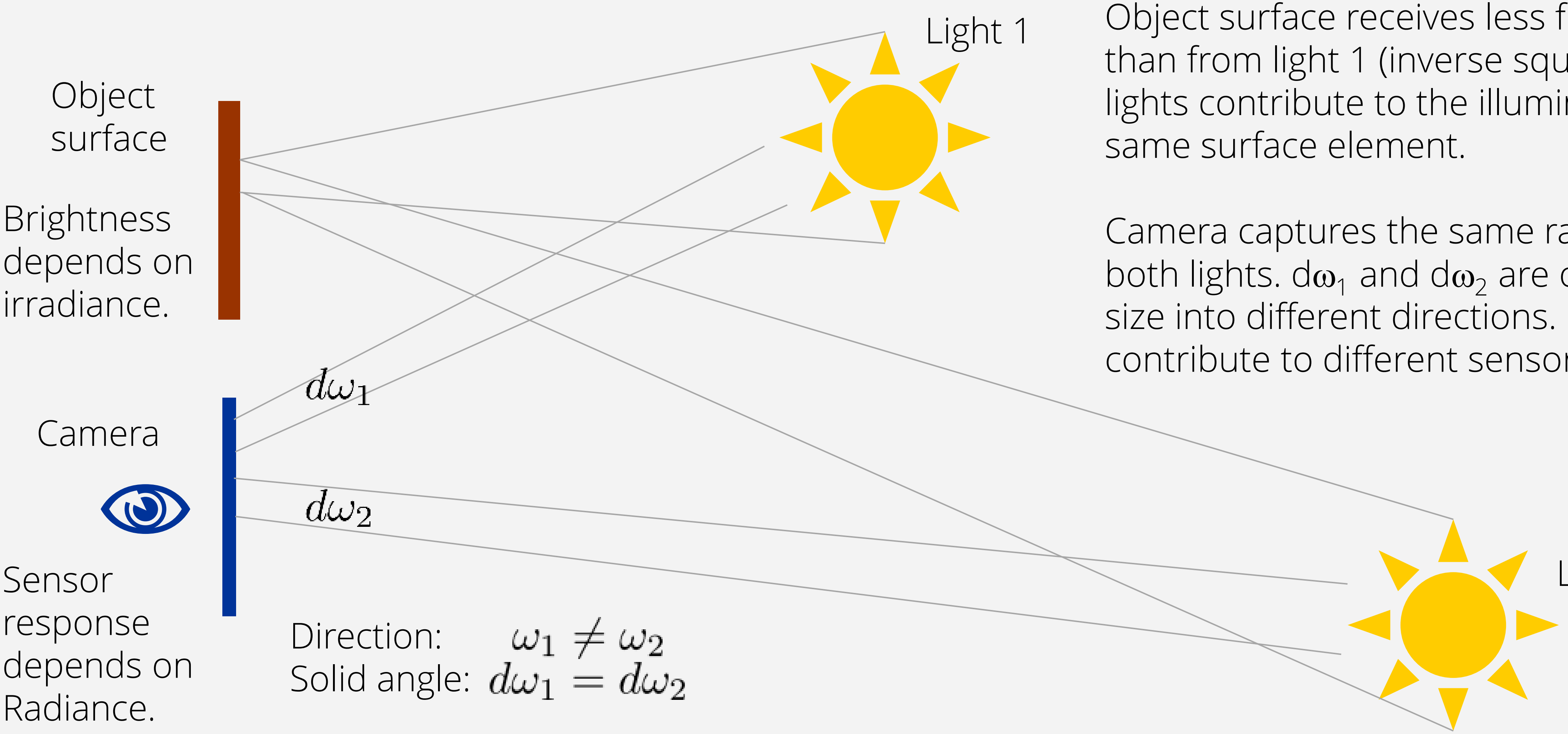


All planes are orthogonal to ω. Thus, cos θ = 1 for all planes.

Irradiance and Radiance

- **Illumination strength at a surface** can be characterized by **irradiance** (flux per area)
  - Depends quadratically on the distance between surface and light source
- **Illumination strength at a sensor element** can be characterized by **radiance** (flux per area per solid angle)
  - Does not depend on the distance between surface and sensor

Irradiance and Radiance



Object surface receives less flux from light 2 than from light 1 (inverse square law). Both lights contribute to the illumination of the same surface element.

Camera captures the same radiance for both lights. $d\omega_1$ and $d\omega_2$ are of the same size into different directions. Light 1 and 2 contribute to different sensor elements.

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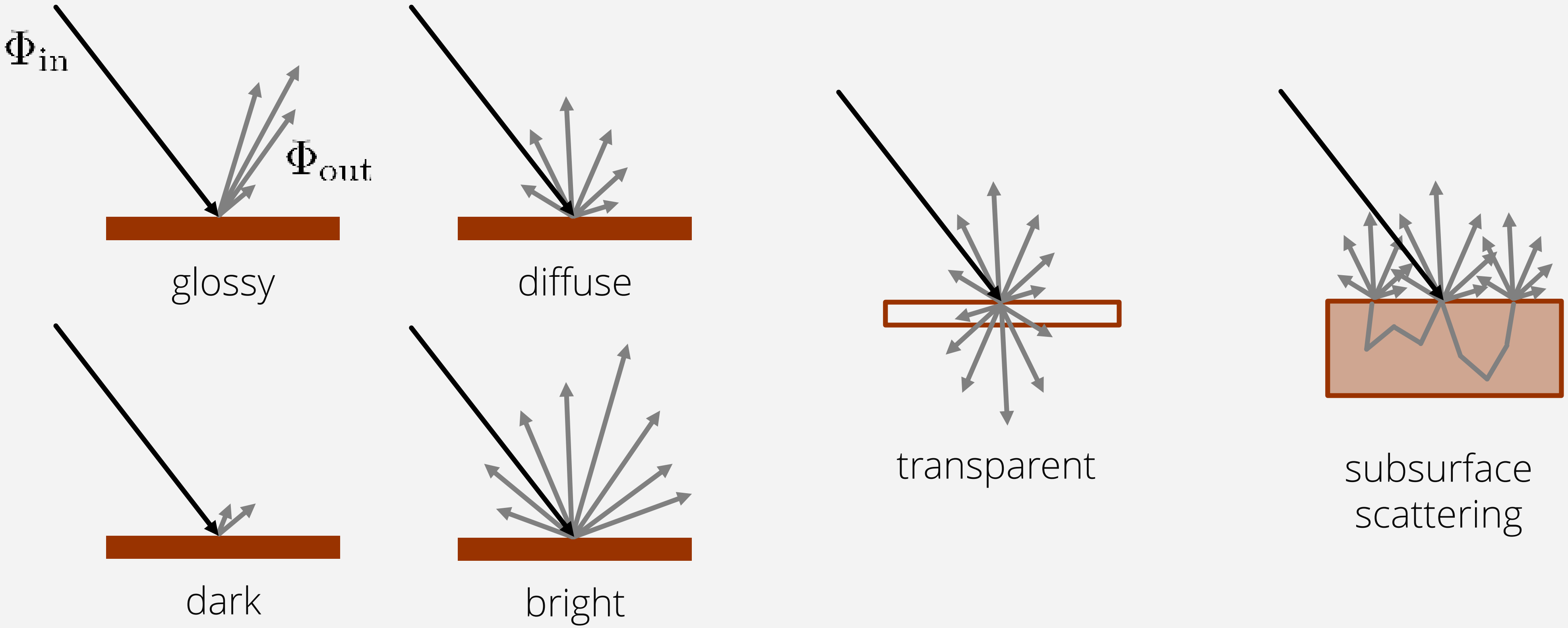
Materials

– Can be described by **relating incident and exitant flux**



Light Interaction at a Surface

- Incident radiance $L_i(p,\omega_i)$ at position p from direction ω_i induces irradiance at p:

  $$dE_i(p,\omega_i) = L_i(p,\omega_i)\cos\theta_i d\omega_i$$

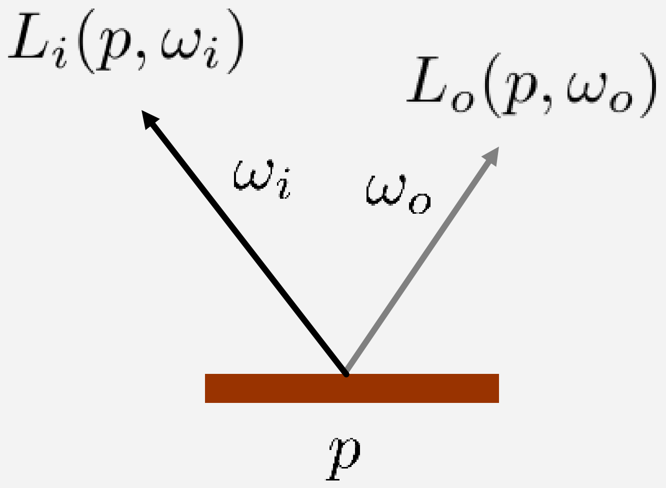
- Flux is partially absorbed:

  $$dB_i(p,\omega_i) = \rho(p)dE_i(p,\omega_i)$$

  $0 \leq \rho \leq 1$ is a reflectance coefficient.

- Reflected flux into direction ω_o

  $$dL_o(p,\omega_o) \sim dB_i(p,\omega_i) \sim dE_i(p,\omega_i)$$



ω_i represents the direction of the incident radiance. Per definition, all directions point away from the surface. I.e., incident radiance travels along $-\omega_i$.

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BRDF Definition

- For all pairs of directions ω_i and ω_o, the ratio of outgoing radiance towards ω_o and irradiance due to incoming radiance from ω_i is referred to as BRDF: $f_r(p, \omega_i, \omega_o) = \frac{dL_o(p,\omega_o)}{dE_i(p,\omega_i)}$
- BRDF typically depends on a position and two directions.
  - Directions form a solid angle of 2π for opaque surfaces and 4π for transparent surfaces
  - Various variants. E.g., BRDF can depend on two positions for subsurface scattering $f_r(p_i, p_o, \omega_i, \omega_o) = \frac{dL_o(p_o,\omega_o)}{dE_i(p_i,\omega_i)}$

BRDF Application

- Relation between irradiance and exitant radiance

$$dL_o(p,\omega_o) = f_r(p,\omega_i,\omega_o)dE_i(p,\omega_i)$$

The portion of light from an incoming direction that is scattered in an outgoing direction

- Irradiance is induced by radiance

$$dL_o(p,\omega_o) = f_r(p,\omega_i,\omega_o)L_i(p,\omega_i)\cos\theta_i d\omega_i$$

- Integration over the hemisphere ⇨ reflectance equation

$$L_o(p,\omega_o) = \int_{2\pi} f_r(p,\omega_i,\omega_o)L_i(p,\omega_i)\cos\theta_i d\omega_i$$

- Reflectance equation establishes a relation between incident and exitant radiance

BRDF Properties

- Definition: $f_r(\omega_i, \omega_o) = \frac{dL_o(\omega_o)}{dE_i(\omega_i)} = \frac{dL_o(\omega_o)}{L_i(\omega_i) \cdot \cos\theta_i \cdot d\omega_i}$
- Positive: $f_r(\omega_i, \omega_o) \geq 0$
- Helmholtz reciprocity: $f_r(\omega_o, \omega_i) = f_r(\omega_i, \omega_o)$
  - Incident and exitant radiance can be reversed
- Energy conservation: $\forall \omega_i : \int_{2\pi^+} f_r(\omega_i, \omega_o) \cos\theta_o \mathrm{d}\omega_o \leq 1$
- Linearity
  - If a material is defined as a sum of BRDFs, the contributions of the BRDFs are added for the total outgoing radiance
  - $\int (f_{r,1} + f_{r,2}) L_i \cos\theta_i d\omega_i = \int f_{r,1} L_i \cos\theta_i d\omega_i + \int f_{r,2} L_i \cos\theta_i d\omega_i$

BRDF Materials

- Diffuse

$$f_{r,d}(\omega_i, \omega_o) = \frac{\rho}{\pi}$$

- Mirror

$$f_{r,m}(\omega_i, \omega_o) = \rho \; \frac{1}{\cos\theta_i \sin\theta_i} \; \delta(\theta_o - \theta_i) \; \delta(\phi_o \pm \pi - \phi_i)$$

- Specular

$$f_{r,s}(\omega_i, \omega_o) = \rho \; ((2(n \cdot \omega_i) \cdot n - \omega_i) \cdot \omega_o)^e$$

n, ω_i, ω_o are represented with 3D normalized vectors

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BRDF for Diffuse Reflecting Material

- Illumination $L_i(\omega_i)$
- Induced surface irradiance $dE_i(\omega_i) = L_i(\omega_i) \cdot \cos\theta_i \cdot d\omega_i$
- Overall irradiance $E = \int_{2\pi} L_i(\omega_i) \cdot \cos\theta_i \cdot d\omega_i$
- Partially absorbed. Resulting radiosity

$$B = \rho \cdot E = \int_{2\pi} \rho \cdot L_i(\omega_i) \cdot \cos\theta_i \cdot d\omega_i \qquad 0 \leq \rho \leq 1$$

ρ - reflectance

$$B = \int_{2\pi} L_o(\omega_o) \cdot \cos\theta_o \cdot d\omega_o = L_o \cdot \int_{2\pi} \cos\theta_o d\omega_o = L_o \cdot \pi$$

see next slide

$$\rho \cdot E = \pi \cdot L_o \qquad L_o = \frac{\rho}{\pi} E = \int_{2\pi} \frac{\rho}{\pi} \cdot L_i(\omega_i) \cdot \cos\theta_i \cdot d\omega_i$$

$$\Rightarrow f_{r,d}(\omega_i, \omega_o) = \frac{\rho}{\pi}$$

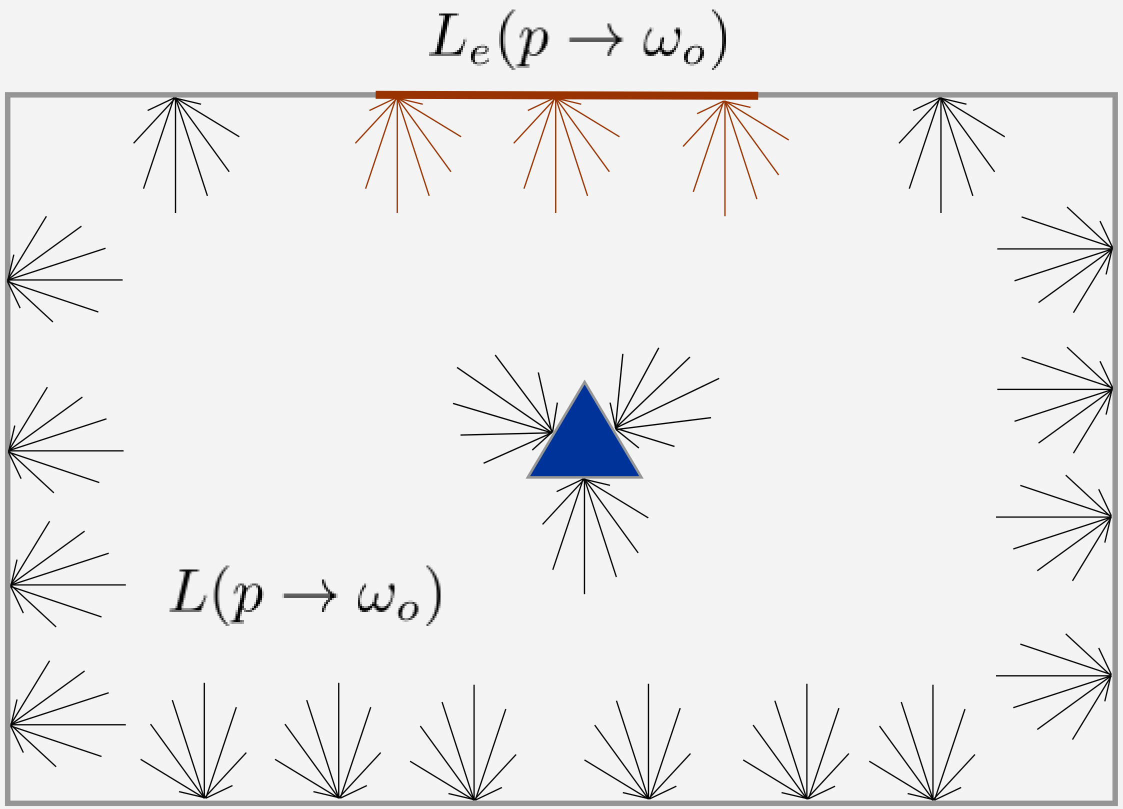
BRDF is constant for diffuse reflecting material

Outline

- Introduction
- Ray-object intersections
- Light
- Materials
- Radiosity
- Stochastic Raytracing

Solution of the Rendering Equation

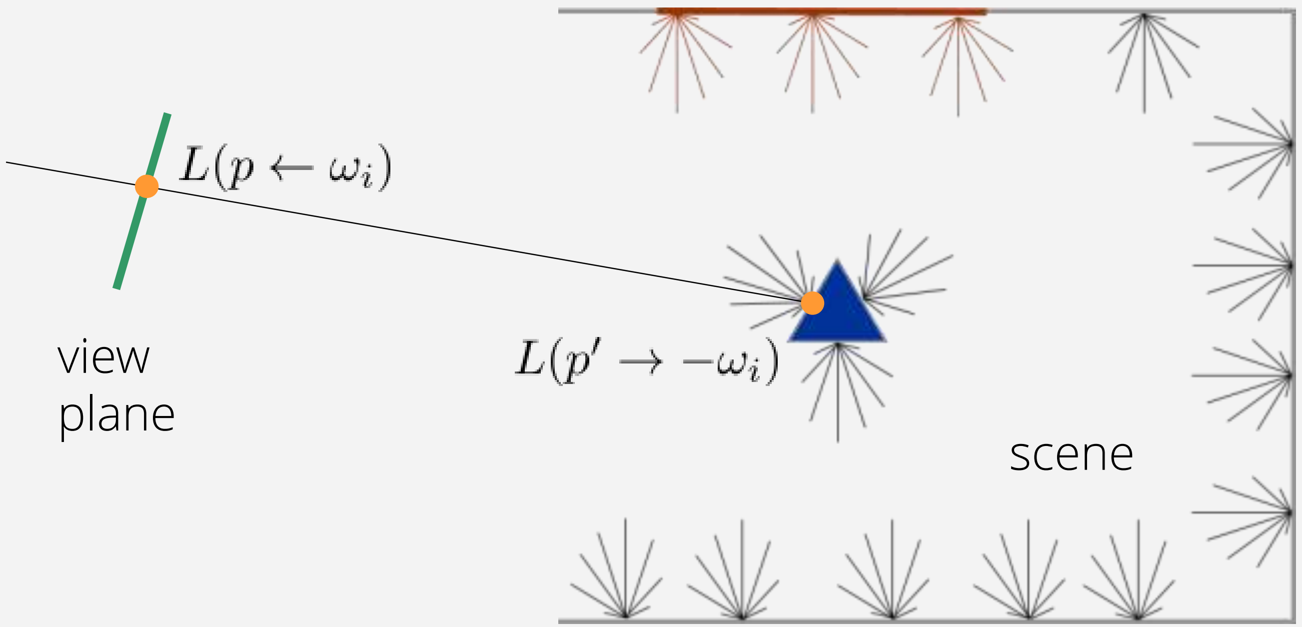
– Exitant radiances from all scene points into all directions



Cornell box

Rendering of the Solution

- At an arbitrarily placed and oriented sensor
  - Cast a ray through position p in an image plane into direction ω_i
  - Lookup $L(p \leftarrow \omega_i) = L(r_c(p, \omega_i) \rightarrow -\omega_i) = L(p' \rightarrow -\omega_i)$



Simplified Setting

- Lambertian material
  - Exitant radiance independent from direction
  - Radiance into arbitrary direction can be computed from radiosity $L(p \rightarrow \omega_o) = \frac{B(p)}{\pi}$
- Discretized scene representation with faces, e.g., triangles
  - Assume constant radiosity per face

⇨ Problem is simplified to n radiosity values for n faces

⇨ n instances of the rendering equation govern the solution

Radiosity Integral Equation

- Rendering equation

$$L(p \to \omega_o) = L_e(p \to \omega_o) +$$
$$\int_S f_r(p, \omega_i \leftrightarrow \omega_o) L(x \to -\omega_i) V(p,x) \frac{\cos(\omega_i, n_p)\cos(-\omega_i, n_x)}{r_{px}^2} dA_x$$

- Radiance can be computed from radiosity for Lambertian surfaces: $L(p \to \omega_o) = \frac{B(p)}{\pi}$
- Radiosity equation

$$B(p) = B_e(p) + \int_S f_r(p, \omega_i \leftrightarrow \omega_o) B(x) V(p,x) \frac{\cos(\omega_i, n_p)\cos(-\omega_i, n_x)}{r_{px}^2} dA_x$$

$$B(p) = B_e(p) + \frac{\rho(p)}{\pi} \int_S B(x) V(p,x) \frac{\cos(\omega_i, n_p)\cos(-\omega_i, n_x)}{r_{px}^2} dA_x$$

Constant BRDF for Lambertian surfaces

Discretization of the Radiosity Equation

– Continuous form, per surface position

$$B(p) = B_e(p) + \frac{\rho(p)}{\pi} \int_S B(x) V(p,x) \frac{\cos(\omega_i, n_p)\cos(-\omega_i, n_x)}{r_{px}^2} dA_x$$

– Discretized form, per face / triangle

Finite Element Method

$$B_i = B_{ei} + \sum_j \rho_i F_{ij} B_j$$

$$B_i - \sum_j \rho_i F_{ij} B_j = B_{ei}$$

– B_{ei} is a source, i.e. the emitted radiosity at face i

– B_i, B_j are unknown radiosities at faces i and j

– ρ_i, F_{ij} are known coefficients

System of Linear Equations

$$
\begin{pmatrix}
1-\rho_1 F_{11} & -\rho_1 F_{12} & \dots & -\rho_1 F_{1n} \\
-\rho_2 F_{21} & 1-\rho_2 F_{22} & \dots & -\rho_2 F_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-\rho_n F_{n1} & \dots & -\rho_n F_{nn-1} & 1-\rho_n F_{nn}
\end{pmatrix}
\begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{pmatrix}
=
\begin{pmatrix} B_{e1} \\ B_{e2} \\ \vdots \\ B_{en} \end{pmatrix}
$$

Matrix with known coefficients, reflectances and form factors. Indirect illumiation. Describes, how faces illuminate each other.

Unknown radiosities.

Known source terms. Direct illumi-nation.

Solving the Linear System

- Typically with iterative schemes, e.g. relaxed Jacobi
  - Initialize, e.g., $B_i^0 = 0$ Superscript indicates solver iteration
  - Iteratively update $B_i^{l+1} = B_i^l + \frac{\lambda_i}{1-\rho_i F_{ii}}(B_{ei} - (B_i^l - \sum_j \rho_i F_{ij} B_j^l))$

    λ_i is a user-defined parameter that governs the solver convergence
- Intuition
  - Changes from B_i^l to B_i^{l+1} are proportional to $B_{ei} - (B_i^l - \sum_j \rho_i F_{ij} B_j^l)$
  - If $B_{ei} - (B_i^l - \sum_j \rho_i F_{ij} B_j^l) = 0$, i.e. $B_i^l - \sum_j \rho_i F_{ij} B_j^l = B_{ei}$, the solver has converged and $B_i^{l+1} = B_i^l$

Outline

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Concept

- Approximately evaluate the integral $\int_\Omega f_r(p, \omega_i \leftrightarrow \omega_o) L(p \leftarrow \omega_i) \cos(\omega_i, n_p) d\omega_i$ by
  - Tracing rays into randomly sampled 2D directions
  - Computing the incoming radiances
- Integral is approximated with $\sum_i f_r(p, \omega_i \leftrightarrow \omega_o) L(p \leftarrow \omega_i) \cos(\omega_i, n_p) \Delta\Omega_i$
  - 2 dimensional sample directions $\omega_i = (\theta_i, \phi_i)$
  - $\Delta\Omega_i$ is an approximation of the solid angle of sample direction $\omega_i = (\theta_i, \phi_i)$

Properties

- Benefits
  - Processes only evaluations of the integrand at arbitrary surface points in the domain
  - Appropriate for integrals of arbitrary dimensions
  - Allows for non-uniform sample patterns / adaptive sample sizes
  - Works for a large variety of integrands, e.g., it handles discontinuities

Properties

- Drawbacks
  - Using n samples, the scheme converges to the correct result with O $(n^{½})$
  - I.e., to half the error, 4n samples are required
  - Errors are perceived as noise, i.e. pixels are arbitrarily too bright or dark (due to the erroneous approximation of the sample size)
  - Evaluation of the integrand at a point is expensive (ray intersections tests)

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Monte Carlo Estimator - Non-uniform Random Variables

- PDF $p(x)$
- Estimator $F_N = \frac{1}{N}\sum_{i=1}^{N}\frac{f(X_i)}{p(X_i)}$
- Integral
  - $\int_a^b f(x)\mathrm{d}x \approx \frac{1}{N}\sum_{i=1}^{N}\frac{f(X_i)}{p(X_i)} = \sum_{i=1}^{N} f(X_i)\frac{1}{N\ p(X_i)}$
  - Function value $f(X_i)$
  - Approximate sample size $\frac{1}{N\ p(X_i)}$

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Monte Carlo Estimator - Integration over a Hemisphere

- Approximate computation of the irradiance at a point

$$E_i(p) = \int_{2\pi^+} L_i(p,\omega)\cos\theta d\omega$$
$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} L_i(p,\theta,\phi)\cos\theta\sin\theta \mathrm{d}\theta\mathrm{d}\phi$$

- Estimator $F_N = \frac{1}{N}\sum_{i=1}^{N}\frac{f(X_i)}{p(X_i)} = \frac{1}{N}\sum_{i=1}^{N}\frac{L_i(p,\theta_i,\phi_i)\cos\theta_i\sin\theta_i}{p(\theta_i,\phi_i)}$
- Choosing a PDF This flexibility is an important aspect of Monte Carlo integration.
  - Should be similar to the shape of the integrand
  - As incident radiance is weighted with cos θ, it is appropriate to generate more samples close to the top of the hemisphere
  - $p(\theta,\phi) \propto \cos\theta$

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Monte Carlo Estimator - Integration over a Hemisphere

- Probability distribution

$$\int_{2\pi^+} c\, p(\omega) \mathrm{d}\omega = 1$$

$$\int_0^{2\pi} \int_0^{\frac{\pi}{2}} c\ \cos\theta \sin\theta\ \mathrm{d}\theta\ \mathrm{d}\phi = 1$$

$$c\ \frac{2\pi}{1+1} = 1$$

$$c = \frac{1}{\pi}$$

$$p(\theta, \phi) = \frac{\cos\theta \sin\theta}{\pi}$$

- Estimator

$$F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{L_i(p, \theta_i, \phi_i) \cos\theta_i \sin\theta_i}{p(\theta_i, \phi_i)}$$

$$= \frac{\pi}{N} \sum_{i=1}^{N} L_i(p, \theta_i, \phi_i) \qquad \approx \int_0^{2\pi} \int_0^{\frac{\pi}{2}} L_i(p, \theta, \phi) \cos\theta \sin\theta \mathrm{d}\theta \mathrm{d}\phi$$

If θ and ϕ are sampled according to PDF $p(\theta, \phi)$

Monte Carlo Estimator - Integration over a Hemisphere

- Integral $\int_0^{2\pi} \int_0^{\frac{\pi}{2}} L_i(p, \theta, \phi) \cos\theta \sin\theta \mathrm{d}\theta \mathrm{d}\phi$
- PDF $p(\theta, \phi) = \frac{\cos\theta \sin\theta}{\pi}$
- Estimator $\frac{\pi}{N} \sum_{i=1}^{N} L_i(p, \theta_i, \phi_i)$
  $= \sum_{i=1}^{N} L_i(p, \theta_i, \phi_i) \cos\theta_i \sin\theta_i \frac{\pi}{N \cos\theta_i \sin\theta_i}$
- Function value $L_i(p, \theta_i, \phi_i) \cos\theta_i \sin\theta_i$ for direction (θ_i, ϕ_i)
- Approximate sample size / solid angle $\frac{\pi}{N \cos\theta_i \sin\theta_i}$

Monte Carlo Integration - Steps

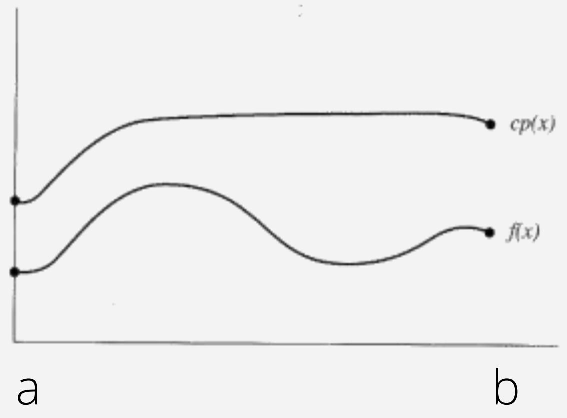
- Choose an appropriate probability density function
- Generate random samples according to the PDF
- Evaluate the function for all samples
- Accumulate sample values weighted with their approximate sample size

Inversion Method

- P and P^{-1} are continuous functions
- Start with the desired PDF $p(x)$
- Compute $P(x) = \int_0^x p(x')\mathrm{d}x'$
- Compute the inverse $P^{-1}(x)$
- Obtain a uniformly distributed variable ξ
- Compute $X_i = P^{-1}(\xi)$ which adheres to $p(x)$

Rejection Method

- Sample generation
  - Generate a uniform random sample $0 \leq \xi < 1$
  - Generate a sample X according to $p(x)$
  - Accept X if $\xi \cdot c \cdot p(X) \leq f(X)$



[Pharr, Humphreys]