Computer Graphics
Particle Fluids

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Course Topics

– Rendering
  – What is visible at a sensor?
    – Ray casting
    – Rasterization / Depth test
  – Which color does it have?
    – Phong

– Modeling
  – Polynomial curves and surfaces

– Simulation
  – Particle fluids
Outline

- Particle simulation
- Particle motion
- Particle forces in a fluid
- Smoothed Particle Hydrodynamics SPH
- SPH for particle fluids
- Neighbor search
- Boundary handling
- Visualization
- Outlook
Particle Simulation

Fluid / Elastic object / Rigid object

Set of parcels

\( x_i^t \)

\( v_i^t \)

Positions and velocities of parcels \( i \) over time \( t \)
Fluid and Solid Parcels
Concept
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Particle Quantities

- Mass \( m \in \mathbb{R} \)
- Position \( \mathbf{x} \in \mathbb{R}^3 \)
- Velocity \( \mathbf{v} \in \mathbb{R}^3 \)
- Force \( \mathbf{F} \in \mathbb{R}^3 \)
- Acceleration \( \mathbf{a} = \frac{\mathbf{F}}{m} \in \mathbb{R}^3 \)
Time Discretization

- Quantities are considered at discrete time points

- Particle simulations are concerned with the computation of unknown future particle quantities $x^{t+h}, v^{t+h}$ from known current information $x^t, v^t, a^t$
  - Where is the parcel? Which velocity does it have?

$h$ is the time step.
Governing Equations

- Newton’s Second Law, motion equation
  \[ \frac{dv^t}{dt} = a^t \quad \frac{dx^t}{dt} = v^t \]
- Ordinary differential equations ODE
- Describe the behavior of \( x^t \) and \( v^t \) in terms of their time derivative
- Numerical integration is employed to approximatively solve the ODEs, i.e. to approximate the unknown functions \( x^t \) and \( v^t \)
Initial Value Problem

– Functions $x^t$ and $v^t$ represent the particle motion
– Initial values $x^t$ and $v^t$ are given
– First-order differential equations are given
  \[
  \frac{dx^t}{dt} = v^t \quad \frac{dv^t}{dt} = a^t
  \]
– How to estimate $x^{t+h}$ and $v^{t+h}$?
Explicit Euler

- Governing equations
  \[ \frac{dx^t}{dt} = v^t \quad \frac{dv^t}{dt} = a^t \]

- Initialization \( x^t = x^{\text{init}} \), \( v^t = v^{\text{init}} \), \( a^t \), \( h \)

- Explicit Euler update
  \[ x^{t+h} = x^t + h \frac{dx^t}{dt} + O(h^2) = x^t + hv^t + O(h^2) \]
  \[ v^{t+h} = v^t + h \frac{dv^t}{dt} + O(h^2) = v^t + ha^t + O(h^2) \]

  Taylor approximation
Alternative Updates, e.g. Verlet

- Taylor approximations of $x^{t+h}$ and $x^{t-h}$
  
  $x^{t+h} = x^t + hv^t + \frac{h^2}{2} a^t + \frac{h^3}{6} \frac{d^3x^t}{dt^3} + O(h^4)$

  $x^{t-h} = x^t - hv^t + \frac{h^2}{2} a^t - \frac{h^3}{6} \frac{d^3x^t}{dt^3} + O(h^4)$

- Adding both approximations leads to the position update
  
  $x^{t+h} = 2x^t - x^{t-h} + h^2 a^t + O(h^4)$

- Velocity update, e.g.
  
  $v^{t+h} = \frac{x^{t+h} - x^t}{h} + O(h)$
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- Force $\mathbf{F} \in \mathbb{R}^3$
- Acceleration $\mathbf{a} = \frac{\mathbf{F}}{m} \in \mathbb{R}^3$
- Density $\rho \in \mathbb{R}$
- Pressure $p \in \mathbb{R}$
Governing Equations for a Fluid

- Particle positions $\mathbf{x}_i^t$ and the respective attributes are advected with the local fluid velocity $\mathbf{v}_i^t$
  \[ \frac{d\mathbf{x}_i^t}{dt} = \mathbf{v}_i^t \]

- Time rate of change of the velocity $\mathbf{v}_i^t$ of an advected sample is governed by the Lagrange form of the Navier-Stokes equation
  \[ \frac{d\mathbf{v}_i^t}{dt} = -\frac{1}{\rho_i^t} \nabla p_i^t + \nu \nabla^2 \mathbf{v}_i^t + \frac{F_{i,\text{other}}^t}{m_i} \]

Accelerations
Accelerations in a Fluid

- $-\frac{1}{\rho_i^t} \nabla p_i^t$: Acceleration due to pressure differences
  - Preserves the fluid rest density $\rho_i^0$
  - Acts in normal direction at the surface of the fluid element
  - Small and preferably constant density deviations are important for high-quality simulation
Accelerations in a Fluid

- $\nu \nabla^2 v^t_i$: Acceleration due to friction forces between particles with different velocities
  - Friction forces act in tangential direction at fluid elements
- $F_{i,t,\text{other}}^{\text{other}} / m_i$: E.g., gravity
Acceleration Terms – 3D

– Volume preservation

\[-\frac{1}{\rho} \nabla p = -\frac{1}{\rho} \left( \begin{array}{c} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{array} \right)\]

– Viscosity

\[\nu \nabla^2 \mathbf{v} = \nu \nabla \cdot (\nabla \mathbf{v}) = \nu \nabla \cdot \left( \begin{array}{ccc} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{array} \right) = \nu \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} + \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \]
Accelerations - Illustration

Volume preservation
External
Viscosity

Volume preservation
External
Viscosity
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Smoothed Particle Hydrodynamics SPH

- Interpolates quantities at arbitrary positions and approximates the spatial derivatives with a finite number of samples, i.e. adjacent particles
- SPH in a Lagrangian fluid simulation
  - Fluid is represented with particles
  - Particle positions and velocities are governed by $\frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i$ and $\frac{d\mathbf{v}_i}{dt} = -\frac{1}{\rho_i} \nabla p_i + \nu \nabla^2 \mathbf{v}_i + \frac{F_{i,\text{other}}}{m_i}$
  - $\rho_i \frac{d\mathbf{v}_i}{dt} - \frac{1}{\rho_i} \nabla p_i$, $\nu \nabla^2 \mathbf{v}_i$ and $\frac{F_{i,\text{other}}}{m_i}$ are computed with SPH
SPH Interpolation

- Quantity $A_i$ at an arbitrary position $x_i$ is approximately computed with a set of known quantities $A_j$ at sample positions $x_j$:

$$A_i = \sum_j A_j \frac{m_j}{\rho_j} W_{ij}$$

- $W_{ij}$ is a kernel function that weights the contributions of sample positions $x_j$ according to their distance to $x_i$. 
SPH Interpolation - Illustrations

\[ W \]

\[ W_{ij} = W(x_j - x_i) = \frac{1}{6h} \left\{ \begin{array}{cl}
(2 - \frac{\|x_j - x_i\|}{h})^3 - 4(1 - \frac{\|x_j - x_i\|}{h})^3 & 0 \leq \frac{\|x_j - x_i\|}{h} < 1 \\
(2 - \frac{\|x_j - x_i\|}{h})^3 & 1 \leq \frac{\|x_j - x_i\|}{h} < 2 \\
0 & \frac{\|x_j - x_i\|}{h} \geq 2
\end{array} \right. \]
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Density

Explicit SPH form

\[ \rho_i = \sum_j \rho_j \frac{m_j}{\rho_j} W_{ij} = \sum_j m_j W_{ij} \]
### Pressure

- Quantifies fluid compression
  - E.g., state equation $p_i = \max \left( k \left( \frac{p_i}{\rho_0} - 1 \right), 0 \right)$
  - Rest density of the fluid $\rho_0$
  - User-defined stiffness $k$

- Pressure acceleration with SPH
  - $\mathbf{a}_i^p = -\frac{1}{\rho_i} \nabla p_i = -\sum_j m_j \left( \frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} \right) \nabla W_{ij}$
  - Accelerates particles from high to low pressure, i.e. from high to low compression to minimize density deviation $\frac{p_i}{\rho_0} - 1$

Pressure values in SPH implementations should always be non-negative.
SPH Discretizations

- Density computation  \( \rho_i = \sum_j m_j W_{ij} \)
- Pressure acceleration  \(- \frac{1}{\rho_i} \nabla p_i = - \sum_j m_j \left( \frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} \right) \nabla W_{ij} \)
- Viscosity acceleration  \( \nu \nabla^2 \mathbf{v}_i = 2\nu \sum_j m_j \frac{1}{\rho_j} \frac{\mathbf{v}_{ij} \cdot \mathbf{x}_{ij}}{\mathbf{x}_{ij} \cdot \mathbf{x}_{ij} + 0.01h^2} \nabla W_{ij} \)
**Simple SPH Fluid Solver**

```
for all particle i do
    find neighbors j

for all particle i do
    \( \rho_i = \sum_j m_j W_{ij} \)
    \( p_i = k\left(\frac{\rho_i}{\rho_0} - 1\right) \)

for all particle i do
    \( a_{i,\text{nonp}} = \nu \nabla^2 v_i + g \)
    \( a_{i,p} = -\frac{1}{\rho_i} \nabla p_i \)
    \( a_i^t = a_{i,\text{nonp}} + a_{i,p} \)

for all particle i do
    \( v_i^{t+\Delta t} = v_i^t + \Delta t a_i^t \)
    \( x_i^{t+\Delta t} = x_i^t + \Delta t v_i^{t+\Delta t} \)
```

Compute adjacent particles for SPH sums

Compute density

Compute pressure

Compute non-pressure accelerations

Compute pressure acceleration

Implicit Euler for position update

Explicit Euler for velocity update
Summary

- Fluid is subdivided into particles
- Navier-Stokes equation states particle accelerations
- SPH states how to approximate these accelerations using adjacent particles (space discretization)
- Fluid solver
  - Compute accelerations
  - Update positions and velocities (time discretization)
  - Accelerations require neighbor search for SPH approximations and density deviation for pressure acceleration
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Uniform Grid - Concept

- Particles are stored in cells
- In $d$-D, potential neighbors in $3^d$ cells are queried to estimate actual neighbors
- Cell size equals the kernel support of a particle
  - Larger cells increase the number of tested particles
  - Smaller cells increase the number of tested cells

Actual neighbors

Potential neighbors

Edge length equals kernel support
Uniform Grid - Implementation

- Compute unique cell identifier per particle
  - Space-filling curves
- Sort particles with respect to cell identifier
  - Particles in the same cell are close to each other
- Map cells to a hash table
  - No explicit representation of the uniform grid
  - Infinitely large grids can be handled
- See *Simulation in Computer Graphics*
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Concept

- Boundaries are sampled with particles that contribute to density, pressure and pressure acceleration of the fluid.

- Boundary handling: How to compute $\rho_i, p_i, p_{ib}, F^p_i$?
Several Layers with Uniform Boundary Samples

- Boundary particles are handled as static fluid samples

\[ \rho_i = \sum_{i_f} m_{i_f} W_{ii_f} + \sum_{i_b} m_{i_b} W_{ii_b} \]

\[ m_i = m_{i_f} = m_{i_b} \]

All samples have the same size, i.e. same mass and rest density

- Pressure acceleration

\[ \mathbf{a}_i^p = -m_i \sum_{i_f} \left( \frac{p_i}{\rho_i^2} + \frac{p_{i_f}}{\rho_{i_f}^2} \right) \nabla W_{ii_f} - m_i \sum_{i_b} \left( \frac{p_i}{\rho_i^2} + \frac{p_{i_b}}{\rho_{i_b}^2} \right) \nabla W_{ii_b} \]

Contributions from fluid neighbors

Contributions from boundary neighbors

Boundary neighbors contribute to the density

All samples have the same size, i.e. same mass and rest density
Pressure at Boundary Samples

- Pressure acceleration at boundaries requires pressure at boundary samples
- Various solutions, e.g. mirroring, extrapolation
- Mirroring
  - Formulation with unknown boundary pressure $p_{ib}$
  - $a^p_i = -m_i \sum_{i_f} \left( \frac{p_i}{\rho_i^2} + \frac{p_{if}}{\rho_{if}^2} \right) \nabla W_{ii_f} - m_i \sum_{i_b} \left( \frac{p_i}{\rho_i^2} + \frac{p_{ib}}{\rho_{ib}^2} \right) \nabla W_{ii_b}$
  - Mirroring of pressure and density from fluid to boundary $p_{ib} = p_i$
  - $a^p_i = -m_i \sum_{i_f} \left( \frac{p_i}{\rho_i^2} + \frac{p_{if}}{\rho_{if}^2} \right) \nabla W_{ii_f} - m_i \sum_{i_b} \left( \frac{p_i}{\rho_i^2} + \frac{p_{ib}}{\rho_{ib}^2} \right) \nabla W_{ii_b}$
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Concept

- Reconstruction and rendering of a triangulated iso-surface
### Iso-Surface Reconstruction – Marching Cubes

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**Input:** Scalar field

**Classification with respect to an iso-value, e.g. 8**

**Output:** Triangulated iso-surface
Initialization

– Density computation at grid points using SPH

$$\rho_i = \sum_j \frac{m_j}{\rho_j} \rho_j W_{ij} = \sum_j m_j W_{ij}$$

Grid sample

Particle samples
Classification

- Inside the fluid: $\rho_i \approx \rho^0$
- Outside: $\rho_i < \rho^0$
- Classification, e.g.
  - $\rho_i \leq 0.5\rho^0 \Rightarrow \text{out}$
  - $\rho_i > 0.5\rho^0 \Rightarrow \text{in}$
Iso-Surface Triangulation

- Generate triangles per voxel, e.g.
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Outlook

- All materials can be simulated with SPH
  - Fluids
  - Viscous fluids
  - Elastic solids
  - Rigid bodies
- ... and their interactions

Valley

up to 38M fluid particles interacting with more than 650 rigid bricks, highly viscous mud and an elastic tree