Course Topics

- Rendering
  - What is visible at a sensor?
    - Ray casting
    - Rasterization / Depth test
  - Which color does it have?
    - Phong
- Modeling
  - Parametric polynomial curves
- Simulation
  - Particle fluids
Outline

- Particle simulation
- Particle motion
- Particle forces in a fluid
- Smoothed Particle Hydrodynamics SPH
- SPH for particle fluids
- Neighbor search
- Boundary handling
- Visualization
- Outlook
Particle Simulation

Fluid / Elastic object / Rigid object

Set of parcels

Positions and velocities of parcels $i$ over time $t$

$x^t_i$  $v^t_i$
Fluid and Solid Parcels
Simulation
Outline

- Particle simulation
- Particle motion
- Particle forces in a fluid
- Smoothed Particle Hydrodynamics SPH
- SPH for particle fluids
- Neighbor search
- Boundary handling
- Visualization
- Outlook
Particle Quantities

- Mass \( m \in \mathbb{R} \)
- Position \( \mathbf{x} \in \mathbb{R}^3 \)
- Velocity \( \mathbf{v} \in \mathbb{R}^3 \)
- Force \( \mathbf{F} \in \mathbb{R}^3 \)
- Acceleration \( \mathbf{a} = \frac{\mathbf{F}}{m} \in \mathbb{R}^3 \)
Time Discretization

- Quantities are considered at discrete time points $t, t + h$

- Particle simulations are concerned with the computation of unknown future particle quantities $x^{t+h}, v^{t+h}$ from known current information $x^t, v^t, a^t$
  - Where is the parcel? Which velocity does it have?

$h$ is the time step.
Governing Equations

- Newton’s Second Law, motion equation
  \[ \frac{dv^t}{dt} = a^t = \frac{F^t}{m} \quad \frac{dx^t}{dt} = v^t \]
- Ordinary differential equations ODE
- Describe the behavior of \( x^t \) and \( v^t \) in terms of their time derivative
- Numerical integration is employed to approximatively solve the ODEs, i.e. to approximate the unknown functions \( x^t \) and \( v^t \)
Initial Value Problem

- Functions $x^t$ and $v^t$ represent the particle motion
- Initial values $x^t$ and $v^t$ are given
- First-order differential equations are given
  \[ \frac{dx^t}{dt} = v^t \quad \frac{dv^t}{dt} = a^t \]
- How to estimate $x^{t+h}$ and $v^{t+h}$?
Explicit Euler

- Governing equations
  \[
  \frac{dx^t}{dt} = v^t, \quad \frac{dv^t}{dt} = \alpha^t
  \]

- Initialization \( x^t = x^{\text{init}}, \quad v^t = v^{\text{init}}, \quad \alpha^t, \quad h \)

- Explicit Euler update
  \[
  x^{t+h} = x^t + h \frac{dx^t}{dt} + O(h^2) = x^t + hv^t + O(h^2)
  \]
  \[
  v^{t+h} = v^t + h \frac{dv^t}{dt} + O(h^2) = v^t + h\alpha^t + O(h^2)
  \]

  Taylor approximation
Alternative Updates, e.g. Verlet

- Taylor approximations of $\mathbf{x}^{t+h}$ and $\mathbf{x}^{t-h}$

\[
\begin{align*}
\mathbf{x}^{t+h} &= \mathbf{x}^t + h\mathbf{v}^t + \frac{h^2}{2} \mathbf{a}^t + \frac{h^3}{6} \frac{d^3 \mathbf{x}}{dt^3} + O(h^4) \\
\mathbf{x}^{t-h} &= \mathbf{x}^t - h\mathbf{v}^t + \frac{h^2}{2} \mathbf{a}^t - \frac{h^3}{6} \frac{d^3 \mathbf{x}}{dt^3} + O(h^4)
\end{align*}
\]

- Adding both approximations leads to the position update

\[
\mathbf{x}^{t+h} = 2\mathbf{x}^t - \mathbf{x}^{t-h} + h^2 \mathbf{a}^t + O(h^4)
\]

- Velocity update, e.g.

\[
\mathbf{v}^{t+h} = \frac{\mathbf{x}^{t+h} - \mathbf{x}^t}{h} + O(h)
\]
Outline

- Particle simulation
- Particle motion
- Particle forces in a fluid
- Smoothed Particle Hydrodynamics SPH
- SPH for particle fluids
- Neighbor search
- Boundary handling
- Visualization
- Outlook
Particle Quantities

- Mass \( m \in \mathbb{R} \)
- Position \( \mathbf{x} \in \mathbb{R}^3 \)
- Velocity \( \mathbf{v} \in \mathbb{R}^3 \)
- Force \( \mathbf{F} \in \mathbb{R}^3 \)
- Acceleration \( \mathbf{a} = \frac{\mathbf{F}}{m} \in \mathbb{R}^3 \)
- Density \( \rho \in \mathbb{R} \)
- Pressure \( p \in \mathbb{R} \)
Governing Equations for a Fluid

- Particle positions $\mathbf{x}_i^t$ and the respective attributes are advected with the local fluid velocity $\mathbf{v}_i^t$

$$\frac{d\mathbf{x}_i^t}{dt} = \mathbf{v}_i^t$$

- Time rate of change of the velocity $\mathbf{v}_i^t$ of an advected sample is governed by the Lagrange form of the Navier-Stokes equation

$$\frac{d\mathbf{v}_i^t}{dt} = -\frac{1}{\rho_i^t} \nabla p_i^t + \nu \nabla^2 \mathbf{v}_i^t + \frac{F_{i,t,\text{other}}}{m_i}$$

Accelerations
Accelerations in a Fluid

- $- \frac{1}{\rho_i} \nabla p_i^t$: Acceleration due to pressure differences
  - Pressure is proportional to compression
  - Particle are accelerated from areas with high pressure / compression to areas with lower pressure / compression
  - Small and preferably constant density deviations / compressions are important for high-quality simulations
Accelerations in a Fluid

- $\nu\nabla^2 \mathbf{v}_i^t$: Acceleration due to friction forces between particles with different velocities
  - Minimizes the difference between a particle velocity and the average velocity of all adjacent particles

- $\frac{F_{i,\text{other}}^t}{m_i}$: E.g., gravity
Acceleration Terms – 3D

- Incompressibility

\[-\frac{1}{\rho} \nabla p = -\frac{1}{\rho} \left( \begin{array}{c} \frac{\partial p}{\partial x_x} \\ \frac{\partial p}{\partial x_y} \\ \frac{\partial p}{\partial x_z} \end{array} \right)\]

- Viscosity

\[\nu \nabla^2 \mathbf{v} = \nu \nabla \cdot (\nabla \mathbf{v}) = \nu \nabla \cdot \left( \begin{array}{ccc} \frac{\partial v_x}{\partial x_x} & \frac{\partial v_x}{\partial x_y} & \frac{\partial v_x}{\partial x_z} \\ \frac{\partial v_y}{\partial x_x} & \frac{\partial v_y}{\partial x_y} & \frac{\partial v_y}{\partial x_z} \\ \frac{\partial v_z}{\partial x_x} & \frac{\partial v_z}{\partial x_y} & \frac{\partial v_z}{\partial x_z} \end{array} \right) = \nu \left( \begin{array}{ccc} \frac{\partial^2 v_x}{\partial x_x^2} + \frac{\partial^2 v_x}{\partial x_y^2} + \frac{\partial^2 v_x}{\partial x_z^2} \\ \frac{\partial^2 v_y}{\partial x_x^2} + \frac{\partial^2 v_y}{\partial x_y^2} + \frac{\partial^2 v_y}{\partial x_z^2} \\ \frac{\partial^2 v_z}{\partial x_x^2} + \frac{\partial^2 v_z}{\partial x_y^2} + \frac{\partial^2 v_z}{\partial x_z^2} \end{array} \right)\]
Outline

- Particle simulation
- Particle motion
- Particle forces in a fluid
- Smoothed Particle Hydrodynamics SPH
- SPH for particle fluids
- Neighbor search
- Boundary handling
- Visualization
- Outlook
Smoothed Particle Hydrodynamics (SPH)

- Interpolates quantities at arbitrary positions and approximates the spatial derivatives with a finite number of samples, i.e., adjacent particles.

- SPH in a fluid simulation:
  - Fluid is represented with particles.
  - Particle positions and velocities are governed by:
    \[
    \frac{dx_i^t}{dt} = \mathbf{v}_i^t \quad \text{and} \quad \frac{d\mathbf{v}_i^t}{dt} = -\frac{1}{\rho_i^t} \nabla p_i^t + \nu \nabla^2 \mathbf{v}_i^t + \frac{\mathbf{F}_{i,\text{other}}}{m_i}
    \]
  - \( \rho_i^t, -\frac{1}{\rho_i^t} \nabla p_i^t, \nu \nabla^2 \mathbf{v}_i^t \) and \( \frac{\mathbf{F}_{i,\text{other}}}{m_i} \) are computed with SPH.
SPH Interpolation

- Quantity $A_i$ at an arbitrary position $\mathbf{x}_i$ is approximately computed with a set of known quantities $A_j$ at sample positions $\mathbf{x}_j$:
  \[ A_i = \sum_j A_j \frac{m_j}{\rho_j} W_{ij} \]

- $W_{ij}$ is a kernel function that weights the contributions of sample positions $\mathbf{x}_j$ their distance to $\mathbf{x}_i$

- Spatial derivatives:
  \[ \nabla A_i = \sum_j A_j \frac{m_j}{\rho_j} \nabla W_{ij} \]
SPH Interpolation - Illustrations

\[ W_{ij} = W(x_j - x_i) = \frac{1}{6h} \left\{ \begin{array}{ll}
\left( 2 - \frac{\|x_j - x_i\|}{h} \right)^3 - 4\left( 1 - \frac{\|x_j - x_i\|}{h} \right)^3 & \text{if } 0 \leq \frac{\|x_j - x_i\|}{h} < 1 \\
0 & \text{if } 1 \leq \frac{\|x_j - x_i\|}{h} < 2 \\
\frac{\|x_j - x_i\|}{h} & \geq 2
\end{array} \right. \]
Outline

- Particle simulation
- Particle motion
- Particle forces in a fluid
- Smoothed Particle Hydrodynamics SPH
- SPH for particle fluids
- Neighbor search
- Boundary handling
- Visualization
- Outlook
Density

- Explicit SPH form

\[ \rho_i = \sum_j \rho_j \frac{m_j}{\rho_j} W_{ij} = \sum_j m_j W_{ij} \]
Pressure

- Quantifies fluid compression
  - E.g., state equation \( p_i = \max(k(\frac{\rho_i}{\rho_0} - 1), 0) \)
  - Rest density of the fluid \( \rho_0 \)
  - User-defined stiffness \( k \)
- Pressure acceleration with SPH
  - \( \mathbf{a}_i^p = -\frac{1}{\rho_i} \nabla p_i = -\sum_j m_j \left( \frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} \right) \nabla W_{ij} \)
  - Accelerates particles from high to low pressure, i.e. from high to low compression to minimize density deviation \( \frac{\rho_i}{\rho_0} - 1 \)

Pressure values in SPH implementations should always be non-negative.
SPH Discretizations

- Density computation \[ \rho_i = \sum_j m_j W_{ij} \]
- Pressure acceleration \[ -\frac{1}{\rho_i} \nabla p_i = -\sum_j m_j \left( \frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} \right) \nabla W_{ij} \]
- Viscosity acceleration \[ \nu \nabla^2 \mathbf{v}_i = 2\nu \sum_j \frac{m_j}{\rho_j} \frac{\mathbf{v}_{ij} \cdot \mathbf{x}_{ij}}{\mathbf{x}_{ij} \cdot \mathbf{x}_{ij} + 0.01h^2} \nabla W_{ij} \]
Simple SPH Fluid Solver

for all particle $i$ do
    find neighbors $j$

for all particle $i$ do
    $\rho_i = \sum_j m_j W_{ij}$
    $p_i = k\left(\frac{p_i}{\rho_0} - 1\right)$

for all particle $i$ do
    Compute density
    Compute pressure

for all particle $i$ do
    $a_{i}^{\text{nonp}} = \nu \nabla^2 \mathbf{v}_i + \mathbf{g}$
    $a_{i}^{\text{p}} = -\frac{1}{\rho_i} \nabla p_i$
    $a_{i}^{t} = a_{i}^{\text{nonp}} + a_{i}^{\text{p}}$

for all particle $i$ do
    Compute non-pressure accelerations
    Compute pressure acceleration

for all particle $i$ do
    $\mathbf{v}_i^{t+\Delta t} = \mathbf{v}_i^{t} + \Delta t a_{i}^{t}$
    $\mathbf{x}_i^{t+\Delta t} = \mathbf{x}_i^{t} + \Delta t \mathbf{v}_i^{t+\Delta t}$

Explicit Euler for velocity update
Implicit Euler for position update
Summary

– Fluid is subdivided into particles
– Navier-Stokes equation states particle accelerations
– SPH states how to approximate these accelerations using adjacent particles (space discretization)
– Fluid solver
  – Compute accelerations
  – Update positions and velocities (time discretization)
  – Accelerations require neighbor search for SPH approximations and density deviation for pressure acceleration
Outline

- Particle simulation
- Particle motion
- Particle forces in a fluid
- Smoothed Particle Hydrodynamics SPH
- SPH for particle fluids
- Neighbor search
- Boundary handling
- Visualization
- Outlook
Uniform Grid - Concept

- Particles are stored in cells
- In $d$-D, potential neighbors in $3^d$ cells are queried to estimate actual neighbors
- Cell size equals the kernel support of a particle
  - Larger cells increase the number of tested particles
  - Smaller cells increase the number of tested cells
Uniform Grid - Implementation

- Compute unique cell identifier per particle
  - Space-filling curves
- Sort particles with respect to cell identifier
  - Particles in the same cell are close to each other
- Map cells to a hash table
  - No explicit representation of the uniform grid
  - Infinitely large grids can be handled
- See Simulation in Computer Graphics
Outline

- Particle simulation
- Particle motion
- Particle forces in a fluid
- Smoothed Particle Hydrodynamics SPH
- SPH for particle fluids
- Neighbor search
- Boundary handling
- Visualization
- Outlook
Concept

- Boundaries are sampled with particles that contribute to density, pressure and pressure acceleration.

$$\begin{align*}
\rho_i &< \rho_0 \\
p_i &= 0 \\
p_{ib} &= 0 \\
F^p_i &= 0
\end{align*}$$

$$\begin{align*}
\rho_i &> \rho_0 \\
p_i &> 0 \\
p_{ib} &> 0 \\
F^p_i &\neq 0
\end{align*}$$

- Boundary handling: How to compute $\rho_i, p_i, p_{ib}, F^p_i$?
Several Layers with Uniform Boundary Samples

- Boundary particles are handled as static fluid samples

\[ \rho_i = \sum_{i_f} m_{i_f} W_{ii_f} + \sum_{i_b} m_{i_b} W_{ii_b} \]

\[ m_i = m_{i_f} = m_{i_b} \]

\[ \rho_i = m_i \sum_{i_f} W_{ii_f} + m_i \sum_{i_b} W_{ii_b} \]

\[ p_i = k\left(\frac{\rho_i}{\rho_0} - 1\right) \]

- Pressure acceleration

\[ \alpha_i^p = -m_i \sum_{i_f} \left(\frac{p_i}{\rho_i^2} + \frac{p_{i_f}}{\rho_{i_f}^2}\right) \nabla W_{ii_f} - m_i \sum_{i_b} \left(\frac{p_i}{\rho_i^2} + \frac{p_{i_b}}{\rho_{i_b}^2}\right) \nabla W_{ii_b} \]

Contributions from fluid neighbors  Contributions from boundary neighbors

Boundary neighbors contribute to the density

All samples have the same size, i.e. same mass and rest density

All samples have the same size, i.e. same mass and rest density
Pressure at Boundary Samples

- Pressure acceleration at boundaries requires pressure at boundary samples
- Various solutions, e.g. mirroring, extrapolation
- Mirroring
  - Formulation with unknown boundary pressure \( p_{ib} \)
    \[
    a^p_i = -m_i \sum_{i_f} \left( \frac{p_i}{\rho_i^2} + \frac{p_{if}}{\rho_{if}^2} \right) \nabla W_{ii_f} - m_i \sum_{i_b} \left( \frac{p_i}{\rho_i^2} + \frac{p_{ib}}{\rho_{ib}^2} \right) \nabla W_{ii_b}
    \]
  - Mirroring of pressure and density from fluid to boundary \( p_{ib} = p_i \)
    \[
    a^p_i = -m_i \sum_{i_f} \left( \frac{p_i}{\rho_i^2} + \frac{p_{if}}{\rho_{if}^2} \right) \nabla W_{ii_f} - m_i \sum_{i_b} \left( \frac{p_i}{\rho_i^2} + \frac{p_i}{\rho_i^2} \right) \nabla W_{ii_b}
    \]
Outline

– Particle simulation
– Particle motion
– Particle forces in a fluid
– Smoothed Particle Hydrodynamics SPH
– SPH for particle fluids
– Neighbor search
– Boundary handling
– Visualization
– Outlook
Concept

- Reconstruction and rendering of a triangulated iso-surface

Akinci et al., ACM Transactions on Graphics, 2012
**Iso-Surface Reconstruction – Marching Cubes**

**Input:** Scalar field

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>12</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
<td>11</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**Classification with respect to an iso-value, e.g. 8**

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>out</td>
<td>out</td>
<td>out</td>
<td>out</td>
</tr>
<tr>
<td>out</td>
<td>in</td>
<td>in</td>
<td>out</td>
</tr>
<tr>
<td>out</td>
<td>out</td>
<td>in</td>
<td>out</td>
</tr>
<tr>
<td>out</td>
<td>out</td>
<td>out</td>
<td>out</td>
</tr>
</tbody>
</table>

**Output:** Triangulated iso-surface

---

University of Freiburg – Computer Science Department – 39
Initialization

- Density computation at grid points using SPH

\[ \rho_i = \sum_j \frac{m_j}{\rho_j} W_{ij} = \sum_j m_j W_{ij} \]

<table>
<thead>
<tr>
<th>Grid sample</th>
<th>Particle samples</th>
</tr>
</thead>
</table>

\[ \rho_i \approx \rho^0 \]

\[ \rho_i < \rho^0 \]
Classification

- Inside the fluid: $\rho_i \approx \rho^0$
- Outside: $\rho_i < \rho^0$
- Classification, e.g.
  
  $$\rho_i \leq 0.5\rho^0 \implies \text{out}$$
  $$\rho_i > 0.5\rho^0 \implies \text{in}$$
Iso-Surface Triangulation

- Generate triangles per cell, e.g.
Outline

- Particle simulation
- Particle motion
- Particle forces in a fluid
- Smoothed Particle Hydrodynamics SPH
- SPH for particle fluids
- Neighbor search
- Boundary handling
- Visualization
- Outlook
Outlook

- All materials can be simulated with SPH
  - Fluids
  - Viscous fluids
  - Elastic solids
  - Rigid bodies
- ... and their interactions

Gissler et al., ACM Transactions on Graphics, 2019