Points on a Line

A line in the Cartesian plane can be written as $ax + by + c = 0$. We consider a point (x, y) on this line and its homogeneous notation (X, Y, W) with $x = \frac{X}{W}$ and $y = \frac{Y}{W}$. This point is on the line, thus: $a\frac{X}{W} + b\frac{Y}{W} + c = 0$. Multiplying with W, we get $aX + bY + cW = 0$.

Interestingly, $aX + bY + cW = 0$ can be checked for all (X, Y, W), even for $W = 0$. E.g., the homogeneous coordinates $(b, -a, 0)$ and $(-b, a, 0)$ are on the line as $ab - ba + c \cdot 0 = -ab + ba + c \cdot 0 = 0$. $(b, -a, 0)$ and $(-b, a, 0)$ can be interpreted as points at infinity on the line $ax + by + c = 0$ or they can be interpreted as the direction of that line.

It can be seen that all lines $ax + by + c = 0$ with fixed a and b, but varying c have the same direction $(b, -a, 0)$ or $(-b, a, 0)$. We have two direction as we can move in two directions on a line. Employing the concept of points at infinity, we can say that parallel lines meet at the same point at infinity: $(b, -a, 0)$ in one direction or $(-b, a, 0)$ in the other direction. These points at infinity only depend on a and b. So, all lines $ax + by + c = 0$ with fixed a and b, but varying c meet at the same two points at infinity.

Points on a Line 2

We consider a line in parametric form $(x(t), y(t)) = \left(t, \frac{-at-c}{b}\right)$. For all t, the Cartesian point $(x(t), y(t))$ fulfills $ax(t) + by(t) + c = 0$, so all points $(x(t), y(t))$ are on that line. One homogeneous notation for $(x(t), y(t))$ is $(x(t), y(t), 1) = \left(t, \frac{-at-c}{b}, 1\right)$. Multiplying with $\frac{b}{t}$, we get $\left(b, -a - \frac{c}{t}, \frac{b}{t}\right)$ with $t \neq 0$. Now, we move to infinity along the line, i.e. $t \to \infty$ or $t \to -\infty$. In this case, the homogeneous form $\left(b, -a - \frac{c}{t}, \frac{b}{t}\right)$ converges to $(b, -a, 0)$ or $(-b, a, 0)$, respectively. Thus, $(b, -a, 0)$ or $(-b, a, 0)$ are points at infinity on the line $ax + by + c = 0$.