Simulation in Computer Graphics

Elastic Solids

Matthias Teschner
Outline

- Introduction
- Elastic forces
- Miscellaneous
- Collision handling
- Visualization
Motivation

- Elastic solids are modeled for particle sets, i.e. elements
- Forces at particles account for resistance to deformation, e.g., stretch, shear, bend, volume change
Motivation

- Element in rest state / undeformed state
  ⇒ No elastic forces
- Element in deformed state
  ⇒ Elastic forces that accelerate particles towards the rest state of an element
Motivation

– Different types of deformation (degrees of freedom) can be derived from the incorporated particles
  – Two particles: stretch, compression
  – Three particles: area, shear
  – More particles: volume, shear
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Overview

- Define elements
- Compute deformation / strain
- Compute stress
- Compute elastic energy
- Compute elastic forces
Elements

- Particles form elements, e.g.
  - Two particles form a line segment
  - Four particles build a tetrahedron
  - ...

Four particles and one tetrahedral element

Element $i$

$X_{i,1}$, $X_{i,2}$, $X_{i,3}$, $X_{i,4}$
Deformation / Strain

- Define a function that describes a deformation of element $i$ based on particle positions: $C_i(x_{i,1}, \ldots, x_{i,n})$
- Undeformed state: $C_i(x_{i,1}, \ldots, x_{i,n}) = 0$
- Deformed state: $C_i(x_{i,1}, \ldots, x_{i,n}) \neq 0$
- Example: relative stretching / compressing of two points $x_1, x_2$ with rest distance $L_i$: 
  $$C_i(x_{i,1}, x_{i,2}) = \frac{1}{L_i}(|x_{i,1} - x_{i,2}| - L_i)$$  
Strain is dimensionless.
Deformation of Line Segments

Five particles and four elements

Exemplary deformation computations for the elements

\[
C_1(x_1, x_2) = \frac{1}{L_1} (|x_1 - x_2| - L_1)
\]
\[
C_2(x_1, x_3) = \frac{1}{L_2} (|x_1 - x_3| - L_2)
\]
\[
C_3(x_1, x_4) = \frac{1}{L_3} (|x_1 - x_4| - L_3)
\]
\[
C_4(x_1, x_5) = \frac{1}{L_4} (|x_1 - x_5| - L_4)
\]
Deformation of a Tetrahedron

\[ C_i(x_{i,1}, \ldots, x_{i,4}) = 0 \quad \text{and} \quad C_i(x_{i,1}, \ldots, x_{i,4}) \neq 0 \]
Stress

- Deformation of an element causes element stress
- Internal pressure (force per area)
- \( S_i(x_{i,1}, \ldots, x_{i,n}) = k_i \ C_i(x_{i,1}, \ldots, x_{i,n}) \)
  - Material stiffness \( k_i \)

\( C_i = 0 \)

Rest state

\( C_i \neq 0 \)

Deformed state

\( S_i \neq 0 \)

Stress due to deformation.

Forces due to stress.
Elastic Energy

- Work that is performed to deform an element is stored as elastic energy:
  - \( E_i(x_{i,1}, \ldots, x_{i,n}) = \frac{1}{2} S_i C_i V_i = \frac{1}{2} k_i C_i^2 V_i \)
  - \( V_i \) is the size (volume / area / length) of an element

- Quantifies the deformation of an element
  - Undeformed state: \( E_i(x_{i,1}, \ldots, x_{i,n}) = 0 \)
  - Deformed state: \( E_i(x_{i,1}, \ldots, x_{i,n}) > 0 \)
**Elastic Forces**

- Accelerate particles from positions with high elastic energy towards positions with low elastic energy
  - Negative spatial gradient of the elastic energy
- Goal: Minimization of elastic energy, i.e. deformation
- For all particles $1 \leq j \leq n$ of an element $i$:

  $$
  F_{i,j}(x_{i,1}, \ldots, x_{i,n}) = -\frac{\partial}{\partial x_{i,j}} E_i(x_{i,1}, \ldots, x_{i,n})
  $$

  $$
  = -k_i V_i C_i(x_{i,1}, \ldots, x_{i,n}) \frac{\partial}{\partial x_{i,j}} C_i(x_{i,1}, \ldots, x_{i,n})
  $$
1D Distance Change

- Two particles $\mathbf{x}_{i,1}, \mathbf{x}_{i,2}$ form an element $i$

$$
\mathbf{x}_{i,1} = \begin{pmatrix} x_{i,1} \\ y_{i,1} \\ z_{i,1} \end{pmatrix} \quad \mathbf{x}_{i,2} = \begin{pmatrix} x_{i,2} \\ y_{i,2} \\ z_{i,2} \end{pmatrix}
$$

- Strain

$$
C^d_i (\mathbf{x}_{i,1}, \mathbf{x}_{i,2}) = \frac{1}{L_i} \left( |\mathbf{x}_{i,1} - \mathbf{x}_{i,2}| - L_i \right)
$$

$$
C^d_i (x_{i,1}, y_{i,1}, z_{i,1}, x_{i,2}, y_{i,2}, z_{i,2})
= \frac{1}{L_i} \left( \sqrt{(x_{i,1} - x_{i,2})^2 + (y_{i,1} - y_{i,2})^2 + (z_{i,1} - z_{i,2})^2} - L_i \right)
$$
1D Distance Change

- Spatial derivatives of the strain

\[
\frac{\partial C_{i}^{d}}{\partial x_{i,1}} = \begin{pmatrix}
\frac{\partial C_{i}^{d}}{\partial x_{i,1}} \\
\frac{\partial C_{i}^{d}}{\partial y_{i,1}} \\
\frac{\partial C_{i}^{d}}{\partial z_{i,1}}
\end{pmatrix} = \frac{1}{L_{i}|x_{i,1} - x_{i,2}|} \begin{pmatrix}
x_{i,1} - x_{i,2} \\
y_{i,1} - y_{i,2} \\
z_{i,1} - z_{i,2}
\end{pmatrix} = \frac{x_{i,1} - x_{i,2}}{L_{i}|x_{i,1} - x_{i,2}|}
\]

\[
\frac{\partial C_{i}^{d}}{\partial x_{i,2}} = -\frac{\partial C_{i}^{d}}{\partial x_{i,1}}
\]
1D Distance Change

- Elastic force at particle $\mathbf{x}_{i,1}$:
  \[
  \mathbf{F}^d_{i,1}(\mathbf{x}_{i,1}, \mathbf{x}_{i,2}) = -k^d_i \quad L_i \quad C^d_i \quad \frac{\partial C^d_i}{\partial \mathbf{x}_{i,1}}
  = -k^d_i \left| \mathbf{x}_{i,1} - \mathbf{x}_{i,2} \right| \frac{L_i}{L_i} \frac{\mathbf{x}_{i,1} - \mathbf{x}_{i,2}}{\left| \mathbf{x}_{i,1} - \mathbf{x}_{i,2} \right|}
  \]

- Elastic force at particle $\mathbf{x}_{i,2}$:
  \[
  \mathbf{F}^d_{i,2}(\mathbf{x}_{i,1}, \mathbf{x}_{i,2}) = -k^d_i \quad L_i \quad C^d_i \quad \frac{\partial C^d_i}{\partial \mathbf{x}_{i,2}}
  = k^d_i \left| \mathbf{x}_{i,1} - \mathbf{x}_{i,2} \right| \frac{L_i}{L_i} \frac{\mathbf{x}_{i,1} - \mathbf{x}_{i,2}}{\left| \mathbf{x}_{i,1} - \mathbf{x}_{i,2} \right|}
  \]
2D Area Change

- Three particles $x_{i,1}, x_{i,2}, x_{i,3}$ form a triangle $i$
- Edges: $e_{i,1} = x_{i,3} - x_{i,1}, e_{i,2} = x_{i,3} - x_{i,2}$
- Strain: $C^a_i = \frac{1}{A_i} \left( \frac{1}{2} |e_{i,1} \times e_{i,2}| - A_i \right)$
- Forces: $F^a_{i,1} = s_i \ e_{i,2} \times t_i$, $F^a_{i,2} = s_i \ t_i \times e_{i,1}$, $F^a_{i,3} = s_i \ t_i \times (e_{i,2} - e_{i,1})$
  
  $s_i = k^a_i \ \frac{C^a_i}{0.5|e_{i,1} \times e_{i,2}|}$
  
  $t_i = e_{i,1} \times e_{i,2}$
3D Volume Change

- Four particles $\mathbf{x}_{i,1}, \mathbf{x}_{i,2}, \mathbf{x}_{i,3}, \mathbf{x}_{i,4}$ form a tetrahedron $i$
- Edges: $\mathbf{e}_{i,1} = \mathbf{x}_{i,2} - \mathbf{x}_{i,1}, \mathbf{e}_{i,2} = \mathbf{x}_{i,3} - \mathbf{x}_{i,1}, \mathbf{e}_{i,2} = \mathbf{x}_{i,4} - \mathbf{x}_{i,1}$
- Strain: $C^v = \frac{1}{V_i} \left( \frac{1}{6} \mathbf{e}_{i,1} (\mathbf{e}_{i,2} \times \mathbf{e}_{i,3}) - V_i \right)$
- Forces $\mathbf{F}_{i,1}^v = k_i^v C_i^v (\mathbf{e}_{i,2} - \mathbf{e}_{i,1}) \times (\mathbf{e}_{i,3} - \mathbf{e}_{i,1})$
  $\mathbf{F}_{i,2}^v = k_i^v C_i^v \mathbf{e}_{i,3} \times \mathbf{e}_{i,2}$
  $\mathbf{F}_{i,3}^v = k_i^v C_i^v \mathbf{e}_{i,1} \times \mathbf{e}_{i,3}$
  $\mathbf{F}_{i,4}^v = k_i^v C_i^v \mathbf{e}_{i,2} \times \mathbf{e}_{i,1}$
Demo

– Tetrahedral mesh with combined constraints $C^d$, $C^a$, $C^v$
Demo

- Volume forces can be used to mimic bending forces in triangulated surfaces.

Two adjacent triangles $i, j$ can be combined to a virtual tetrahedron $k$. A volume change of $k$ corresponds to an angle change between $n_i$ and $n_j$. 
Properties of Elastic Forces

- Preserve linear and angular momentum of an element or of a system of elements

Distance forces change the distance between the two points, but not the linear and angular velocity of the spring.

Volume forces change the volume of the tetrahedron, but not its velocity.
Properties of Elastic Forces

- Sum up to zero for all elements
  - $\sum_{j=1..n} F_{i,j}(x_{i,1}, \ldots, x_{i,n}) = 0$
  - Do not change linear momentum
- Do not cause torque
  - Do not change angular momentum
- Also referred to as internal forces
  - External forces can change linear and angular momentum of an element, e.g. gravitational force

Forces sum up to zero, but change angular momentum of the element $\implies$ no elastic force
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Damping

– Force proportional to a velocity
– Directed in the opposite direction of a velocity
– Models friction
– Improves the stability of a particle system in explicit integration schemes
  – Typically omitted in implicit schemes with artificial damping
Damping

- Particle velocity

\[ F^{\text{damp}}_i = -\gamma \, v_i \]

Damping parameter

- Relative velocity

\[ F^{\text{damp}}_{i,j} = \gamma \left( (v_j - v_i) \frac{x_j - x_i}{\|x_j - x_i\|} \right) \frac{x_j - x_i}{\|x_j - x_i\|} \]

Damping parameter \quad Relative velocity projected onto direction \(x_j - x_i\) \quad Normalized direction
Damping

- Particle velocity
  - External force
  - Affects the global dynamics of a particle system, i.e. slows it down

- Relative velocity
  - Internal force
  - Reduces oscillations / noise
  - Does not affect linear and angular momentum of a particle system
Particle Masses

- Should be proportional to the particle size
  - Discretization should not affect the simulation

- E.g., \( m_i = \rho_i \left( \frac{1}{3} \sum_j A_j \right) \)
  
  Mass density \( \rho_i \) of the material
  
  Particle size \( A_j \)

- Elastic accelerations are accumulated at particles
  
  \( \mathbf{a}_i = \rho_i \left( \frac{1}{3} \sum_j A_j \right) \left( \sum_j \mathbf{F}_{j,i} \right) \) or
  
  \( \mathbf{a}_i = \sum_j \left( \rho_j \frac{1}{3} A_j \mathbf{F}_{j,i} \right) \)
Time Step

- An element should not move farther than its size in one simulation step, e.g. its diameter $d$: $h|v| \leq d$
- Time step limit: $h \leq \frac{d}{|v|}$
- $h = \lambda \frac{d}{|v|}$ with $0 < \lambda \leq 1$
- $\lambda = \frac{h|v|}{d}$ can be interpreted as performance measure
- Time step size is only meaningful if related to the element size
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Context

– Collision of a particle of an elastic solid with a plane
Plane Representation

- In 3D, a plane can be defined with a point $\mathbf{x}_p$ on the plane and a normalized plane normal $\mathbf{n}_{\text{plane}}$.
- The plane is the set of points $\mathbf{x}$ with $\mathbf{n}_{\text{plane}} \cdot (\mathbf{x} - \mathbf{x}_p) = 0$.
- For a point $\mathbf{x}$, the distance to the plane is $d = \mathbf{n}_{\text{plane}} \cdot (\mathbf{x} - \mathbf{x}_p)$. 

![Diagram showing plane representation](image)
Concept

– If a collision is detected, i.e. \( d < 0 \), a collision impulse is computed that prevents the interpenetration of the mass point and the plane.

– We first consider the case of a particle-particle collision with \( \mathbf{n} \) being the normalized direction from \( \mathbf{x}_2 \) to \( \mathbf{x}_1 \).

– The response scheme is later adapted to the particle-plane case.
Coordinate Systems

- Velocities $\mathbf{v}$ before the collision response and $\mathbf{V}$ velocities after the collision response are considered in the coordinate system defined by collision normal $\mathbf{n}$ and two orthogonal normalized tangent axes $\mathbf{t}$ and $\mathbf{k}$.

- E.g.
  \[
  \begin{pmatrix}
  v_{1,n} \\
  v_{1,t} \\
  v_{1,k}
  \end{pmatrix}
  =
  \begin{pmatrix}
  n_x & n_y & n_z \\
  t_x & t_y & t_z \\
  k_x & k_y & k_z
  \end{pmatrix}
  \begin{pmatrix}
  v_{1,x} \\
  v_{1,y} \\
  v_{1,z}
  \end{pmatrix}
  \]

- The velocity $\mathbf{V}$ after the response is transformed back
  \[
  \begin{pmatrix}
  V_{1,x} \\
  V_{1,y} \\
  V_{1,z}
  \end{pmatrix}
  =
  \begin{pmatrix}
  n_x & t_x & k_x \\
  n_y & t_y & k_y \\
  n_z & t_z & k_z
  \end{pmatrix}
  \begin{pmatrix}
  V_{1,n} \\
  V_{1,t} \\
  V_{1,k}
  \end{pmatrix}
  \]
Governing Equations

– Conservation of momentum

\[ m_1 V_{1,n} - m_1 v_{1,n} = P_n \]
\[ m_2 V_{2,n} - m_2 v_{2,n} = -P_n \]
\[ m_1 V_{1,t} - m_1 v_{1,t} = P_t \]
\[ m_2 V_{2,t} - m_2 v_{2,t} = -P_t \]
\[ m_1 V_{1,k} - m_1 v_{1,k} = P_k \]
\[ m_2 V_{2,k} - m_2 v_{2,k} = -P_k \]

– Coefficient of restitution, \( e = 1 \) elastic, \( e = 0 \) inelastic

\[ V_{1,n} - V_{2,n} = -e(v_{1,n} - v_{2,n}) \]

– Friction opposes sliding motion along \( t \) and \( k \)

\[ P_t = \mu P_n \]
\[ P_k = \mu P_n \]
Linear System

\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
-\mu & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\mu & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & m_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & m_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & m_1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & m_2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & m_2 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & m_2 & 0 \\
\end{pmatrix}
\begin{pmatrix}
P_n \\
P_k \\
P_t \\
V_{1,n} \\
V_{1,t} \\
V_{1,k} \\
V_{2,n} \\
V_{2,t} \\
V_{2,k} \\
\end{pmatrix}
= 
\begin{pmatrix}
-e(v_{1,n} - v_{2,n}) \\
0 \\
0 \\
m_1 v_{1,n} \\
m_1 v_{1,t} \\
m_1 v_{1,k} \\
m_2 v_{2,n} \\
m_2 v_{2,t} \\
m_2 v_{2,k} \\
\end{pmatrix}
\]
Solution

\[
\begin{pmatrix}
\frac{m_1 m_2}{m_1 + m_2} & 0 & 0 & -\frac{m_2}{m_1 + m_2} & 0 & 0 & \frac{m_1}{m_1 + m_2} & 0 & 0 \\
\frac{m_1 m_2 \mu}{m_1 + m_2} & 1 & 0 & -\frac{m_2 \mu}{m_1 + m_2} & 0 & 0 & \frac{m_1 \mu}{m_1 + m_2} & 0 & 0 \\
\frac{m_1 m_2}{m_1 + m_2} & 0 & 1 & -\frac{m_2}{m_1 + m_2} & 0 & 0 & \frac{m_1}{m_1 + m_2} & 0 & 0 \\
\frac{m_2}{m_1 + m_2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{m_1 + m_2} & 0 & 0 \\
\frac{m_1 \mu}{m_1 + m_2} & 0 & 0 & -\frac{m_2 \mu}{m_1 (m_1 + m_2)} & 1 & 0 & \frac{m_1 \mu}{m_1 + m_2} & 0 & 0 \\
\frac{m_2 \mu}{m_1 + m_2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{m_1 + m_2} & 0 & 0 \\
\frac{m_1}{m_1 + m_2} & 0 & 0 & -\frac{1}{m_1 + m_2} & 0 & 0 & \frac{1}{m_1 + m_2} & 0 & 0 \\
\frac{m_2}{m_1 + m_2} & 0 & 0 & -\frac{1}{m_2 + m_2} & 0 & 0 & -\frac{m_1 \mu}{m_2 (m_1 + m_2)} & \frac{1}{m_2} & 0 \\
\frac{m_1 \mu}{m_1 + m_2} & 0 & 0 & -\frac{1}{m_2 + m_2} & 0 & 0 & -\frac{m_1 \mu}{m_2 (m_1 + m_2)} & \frac{1}{m_2} & 0 \\
\end{pmatrix} \begin{pmatrix}
-e(v_{1,n} - v_{2,n}) \\
0 \\
0 \\
1 \\
m_1 v_{1,n} \\
m_1 v_{1,t} \\
m_1 v_{1,k} \\
m_2 v_{2,n} \\
m_2 v_{2,t} \\
m_2 v_{2,k} \\
\end{pmatrix} = \begin{pmatrix}
P_n \\
P_k \\
V_{1,n} \\
V_{1,t} \\
V_{1,k} \\
V_{2,n} \\
V_{2,t} \\
V_{2,k} \\
\end{pmatrix}
\]
Particle \mapsto Plane

- Plane has infinite mass and does not move: \( v_2 = V_2 = 0 \)
- Columns 2, 3, 7, 8, 9 do not contribute to the solution
- To solve for the particle velocity \( V_1 \) after collision response, rows 4, 5, 6 have to be considered

\[
\begin{pmatrix}
\frac{m_2}{m_1 + m_2} - \frac{1}{m_1 (m_1 + m_2)} & 0 & 0 \\
\frac{m_2 \mu}{m_1 + m_2} & \frac{1}{m_1 (m_1 + m_2)} & 0 \\
\frac{m_2 \mu}{m_1 + m_2} & \frac{1}{m_1 (m_1 + m_2)} & \frac{1}{m_1}
\end{pmatrix}
\begin{pmatrix}
-e v_{1,n} \\
m_1 v_{1,n} \\
m_1 v_{1,t}
\end{pmatrix}
= 
\begin{pmatrix}
V_{1,n} \\
V_{1,t} \\
V_{1,k}
\end{pmatrix}
\]

- Plane has infinite mass

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
\mu & -\frac{\mu}{m_1} & \frac{1}{m_1} & 0 \\
\mu & -\frac{\mu}{m_1} & \frac{1}{m_1} & 0
\end{pmatrix}
\begin{pmatrix}
-e v_{1,n} \\
m_1 v_{1,n} \\
m_1 v_{1,t} \\
m_1 v_{1,k}
\end{pmatrix}
= 
\begin{pmatrix}
V_{1,n} \\
V_{1,t} \\
V_{1,k}
\end{pmatrix}
\]
Implementation

\[ V_{t,n} = -e v_{t,n} \]
\[ V_{t,t} = v_{t,t} - \mu (e + 1) v_{t,n} \]
\[ V_{t,k} = v_{t,k} - \mu (e + 1) v_{t,n} \]

- \( \mu \) is difficult to handle
- \( |V_{t,t}| \leq |v_{t,t}| \) and \( \text{sign}(V_{t,t}) = \text{sign}(v_{t,t}) \)
  
  should be guaranteed

- \( V_{t,t} = \mu v_{t,t} \quad V_{t,k} = \mu v_{t,k} \quad 0 \leq \mu \leq 1 \)
  
  is a useful simplification
Position Update

– The collision impulse updates the velocity
– However, the point is still in collision ($d < 0$)
– For low velocities, the position update in the following integration step may not be sufficient to resolve the collision
– Therefore, the position should be updated as well, e.g. $x_{t+h} = x_t + d \cdot n$ which projects the point onto the plane
– The position update is not physically-motivated, it just resolves problems due to discrete time steps
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Context

– Geometric combination of
  – A low-resolution tetrahedral mesh for simulation and
  – A high-resolution triangular mesh for visualization
– Supports simplified meshing for geometrically complex surface models
Illustration

- Two representations for simulation and visualization
- Tetrahedral elements with particles $x_i$ are simulated
- Triangulated elements with vertices $x_s$ are visualized
Barycentric Coordinates

- Surface vertex $\mathbf{x}_s$ can be represented with the particles of a tetrahedron
  \[
  \mathbf{x}_s = \mathbf{x}_1 + \alpha_2(\mathbf{x}_2 - \mathbf{x}_1) + \alpha_3(\mathbf{x}_3 - \mathbf{x}_1) + \alpha_4(\mathbf{x}_4 - \mathbf{x}_1)
  \]
  \[
  \mathbf{x}_s = (1 - \alpha_2 - \alpha_3 - \alpha_4)\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_3\mathbf{x}_3 + \alpha_4\mathbf{x}_4
  \]
  \[
  \mathbf{x}_s = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_3\mathbf{x}_3 + \alpha_4\mathbf{x}_4
  \]
  \[
  \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1
  \]
- $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are Barycentric coordinates of $\mathbf{x}_s$ with respect to $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$
Properties

- $0 < \alpha_i < 1$
  $x_s$ is inside the convex combination of $x_1, x_2, x_3, x_4$, i.e. inside the tetrahedron

- $\alpha_i = 0 \lor \alpha_i = 1$
  $x_s$ is on the surface of the tetrahedron

- $\alpha_i < 0 \lor \alpha_i > 1$
  $x_s$ is outside the tetrahedron
\[ \mathbf{x}_s = \mathbf{x}_1 + \alpha_2 (\mathbf{x}_2 - \mathbf{x}_1) + \alpha_3 (\mathbf{x}_3 - \mathbf{x}_1) + \alpha_4 (\mathbf{x}_4 - \mathbf{x}_1) \]

leads to the following system

\[
\begin{pmatrix}
(x_2 - x_1) \\
(x_3 - x_1) \\
(x_4 - x_1)
\end{pmatrix}
\begin{pmatrix}
\alpha_2 \\
\alpha_3 \\
\alpha_4
\end{pmatrix}
= \mathbf{x}_s - \mathbf{x}_1
\]

– Not solvable for degenerated tetrahedra

– \( \alpha_1 \) is computed as \( \alpha_1 = 1 - \alpha_2 - \alpha_3 - \alpha_4 \)
Implementation

– Preprocessing
  – Determine the closest tetrahedron for surface points
  – Compute Barycentric coordinates for surface points with respect to the corresponding tetrahedron

– Simulation step
  – Compute surface-point positions from Barycentric coordinates and the positions of the particles of the corresponding tetrahedron

– Demo