Simulation in Computer Graphics

Stability of Mass-Point Systems

Matthias Teschner

Computer Science Department
University of Freiburg

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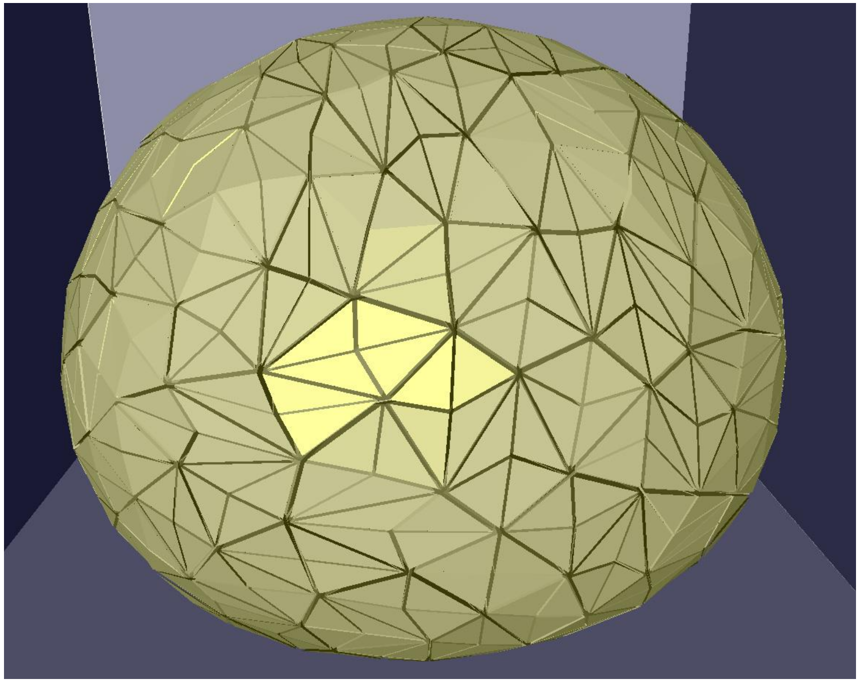
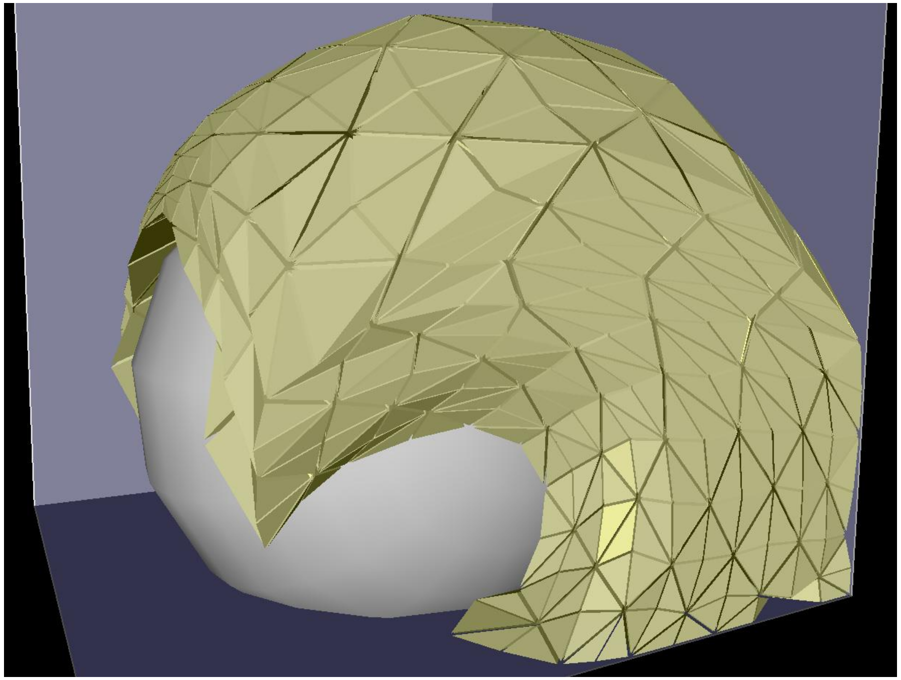
Demos



surface tension vs.
volume preservation



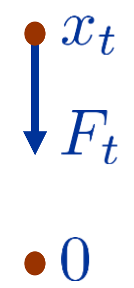
distance preservation vs.
volume preservation

Outline

- motivation
- numerical integration as a transformation
- eigenanalysis
- examples
- discussion

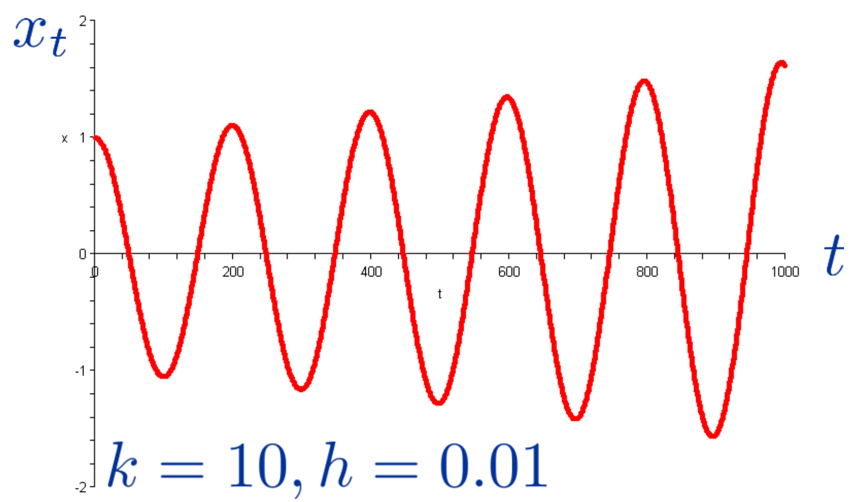
Motivation

- one-dimensional spring with resting length zero
- one end point is fixed at $x = 0$
- for the second point, $m = 1$ and $F_t = -kx_t$



- explicit Euler integration

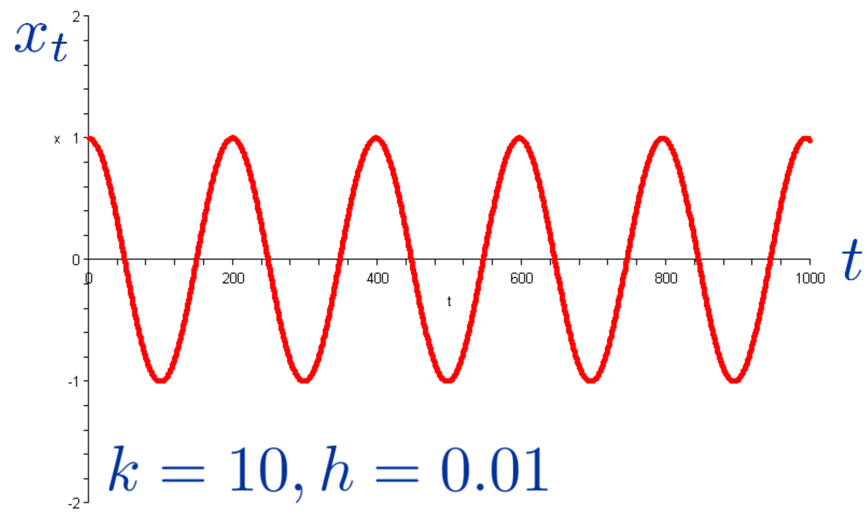
$$
\begin{aligned}
x_{t+h} &= x_t + hv_t \\
v_{t+h} &= v_t - hkx_t
\end{aligned}
$$



Motivation

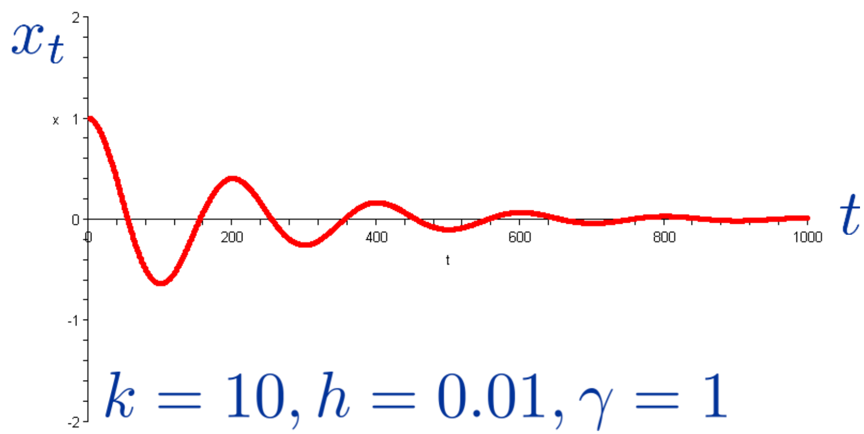
- Euler-Cromer

$$
\begin{aligned}
x_{t+h} &= x_t + h v_{t+h} \\
v_{t+h} &= v_t - h k x_t
\end{aligned}
$$



- explicit Euler (with damping)

$$
\begin{aligned}
x_{t+h} &= x_t + h v_t \\
v_{t+h} &= v_t - h k x_t - h \gamma v_t
\end{aligned}
$$



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State Transformation

- explicit Euler integration

  $$x_{t+h} = x_t + h v_t \quad v_{t+h} = v_t - hkx_t$$

- state vector $\mathbf{u}_t = \begin{pmatrix} x_t & v_t \end{pmatrix}^{\mathrm{T}}$
- integration can be seen as a transformation applied to $\mathbf{u}_t$
- explicit Euler (without damping)

  $$\mathbf{u}_{t+h} = \mathbf{A}\mathbf{u}_t = \begin{pmatrix} 1 & h \\ -hk & 1 \end{pmatrix} \mathbf{u}_t$$

- explicit Euler (with damping)

  $$\mathbf{u}_{t+h} = \mathbf{A}\mathbf{u}_t = \begin{pmatrix} 1 & h \\ -hk & 1 - h\gamma \end{pmatrix} \mathbf{u}_t$$

Approximation Error

- arbitrary integration schemes transform a state $\mathbf{u}$ of a system from time t to time $t+h$
  $\mathbf{u}_{t+h} = \mathbf{A}\mathbf{u}_t$
- successive application of $\mathbf{A}$ results in a sequence
  $\mathbf{u}_t, \mathbf{u}_{t+h}, \mathbf{u}_{t+2h}, \ldots$
- this sequence approximates the correct solution
  $\mathbf{u}_t + \epsilon_t, \mathbf{u}_{t+h} + \epsilon_{t+h}, \mathbf{u}_{t+2h} + \epsilon_{t+2h}, \ldots$
- ϵ_i are errors introduced by the integration scheme

Amplification Matrix

- we have $\mathbf{u}_{t+h} + \epsilon_{t+h} = \mathbf{A}(\mathbf{u}_t + \epsilon_t)$
- if $\mathbf{A}$ is linear
$$\begin{aligned} \mathbf{u}_{t+h} + \epsilon_{t+h} &= \mathbf{A}\mathbf{u}_t + \mathbf{A}\epsilon_t \\ \epsilon_{t+h} &= \mathbf{G}\epsilon_t = \mathbf{A}\epsilon_t \end{aligned}$$
- if $\mathbf{A}$ is non-linear
$$\begin{aligned} \mathbf{u}_{t+h} + \epsilon_{t+h} &= \mathbf{A}(\mathbf{u}_t + \epsilon_t) \approx \mathbf{A}\mathbf{u}_t + \frac{\partial \mathbf{A}\mathbf{u}_t}{\partial \mathbf{u}_t}\epsilon_t \\ \epsilon_{t+h} &= \mathbf{G}\epsilon_t \approx \frac{\partial \mathbf{A}\mathbf{u}_t}{\partial \mathbf{u}_t}\epsilon_t \end{aligned}$$
- $\mathbf{G}$ is the amplification matrix

Stability

- if the repeated multiplication of $\mathbf{G}$ with any previously introduced approximation error ϵ_i is diverging, the integration scheme is unstable
- remember:
  a FDE is stable, if previously introduced errors
  do not grow within a simulation step
- approximation errors are introduced in each integration step, however, if previously introduced errors are not amplified by the integration scheme, then it is stable

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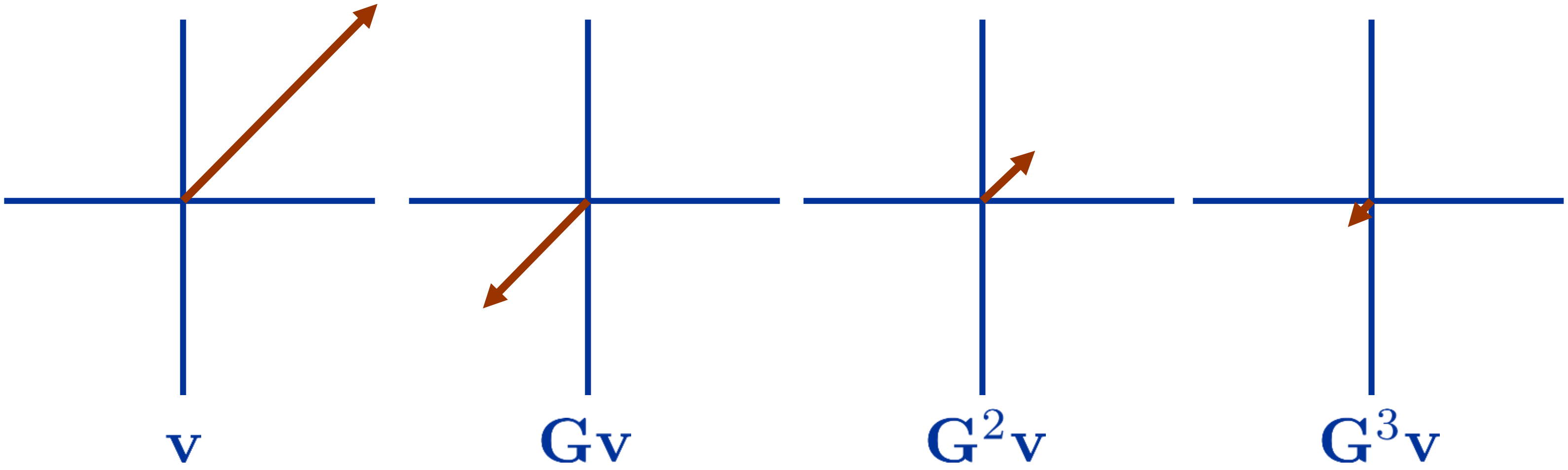
Eigenanalysis

- an eigenvector $\mathbf{v}$ of a matrix $\mathbf{G}$ is a nonzero vector that is scaled by a scalar λ when $\mathbf{G}$ is applied to it
  $\mathbf{G}\mathbf{v} = \lambda\mathbf{v}$
- λ is an eigenvalue of $\mathbf{G}$
- if $\mathbf{v}$ is an eigenvector, $\alpha\mathbf{v}$ is also an eigenvector
  $\mathbf{G}(\alpha\mathbf{v}) = \alpha\mathbf{G}\mathbf{v} = \lambda\alpha\mathbf{v}$

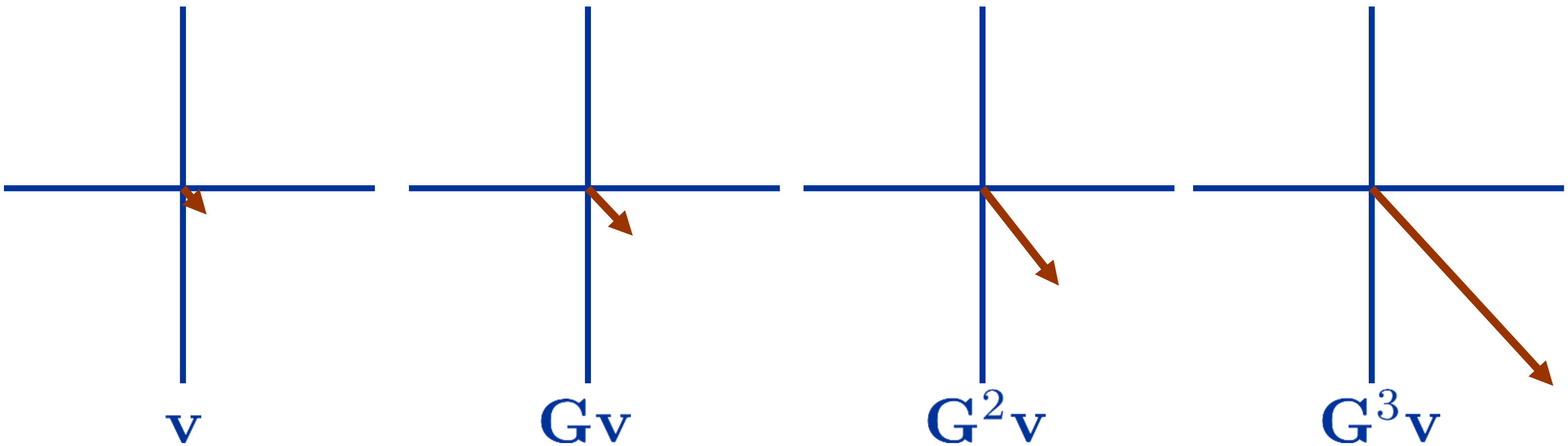
Shrinking Eigenvectors

- matrix $\mathbf{G}$ scales the eigenvector $\mathbf{v}$ with eigenvalue λ
- if $|\lambda| < 1$ then $\mathbf{G}^i\mathbf{v} = \lambda^i\mathbf{v}$
  vanishes for i approaching infinity
- repeated application of $\mathbf{G}$ to $\mathbf{v}$ shrinks $\mathbf{v}$
- example with $\lambda = -0.5$



Growing Eigenvectors

- if $|\lambda| > 1$ then $\mathbf{G}^i\mathbf{v} = \lambda^i\mathbf{v}$
  grows to infinity for i approaching infinity
- repeated application of $\mathbf{G}$ to $\mathbf{v}$ enlarges $\mathbf{v}$
- example with $\lambda = 2$



General Case

- if the magnitude of all eigenvalues is smaller one, all eigenvectors vanish when $\mathbf{G}$ is repeatedly applied
- all vectors that are linear combinations of eigenvectors behave like eigenvectors
- if there exist n linearly independent eigenvectors of $\mathbf{G}$, then all vectors are linear combinations of these eigenvectors $\epsilon = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n$

General Case

- if $\mathbf{G}$ is applied to any vector

$$\mathbf{G}\epsilon = \alpha_1 \mathbf{G}\mathbf{v}_1 + \alpha_2 \mathbf{G}\mathbf{v}_2 + \ldots + \alpha_n \mathbf{G}\mathbf{v}_n$$
$$\mathbf{G}\epsilon = \alpha_1 \lambda_1 \mathbf{v}_1 + \alpha_2 \lambda_2 \mathbf{v}_2 + \ldots + \alpha_n \lambda_n \mathbf{v}_n$$

- then $\mathbf{G}^i\epsilon$ is vanishing if all eigenvalues λ_i are smaller one
- if at least one eigenvalue is larger than one, $\mathbf{G}^i\epsilon$ diverges to infinity

Spectral Radius / Stability

- spectral radius ρ of a matrix $\mathbf{G}$
  $\rho(\mathbf{G}) = \max |\lambda_i|$

- λ_i is an eigenvalue of $\mathbf{G}$
- if the spectral radius of the amplification matrix of an integration scheme is smaller one, the scheme is stable ($\mathbf{G}^i\epsilon$ is vanishing, previous errors are diminished)
- if the spectral radius is larger one,
  the scheme is unstable
  ($\mathbf{G}^i\epsilon$ is diverging, previous errors are amplified)

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Example – Explicit Euler

$$\begin{aligned} x_{t+h} &= x_t + hv_t \\ v_{t+h} &= v_t - hkx_t \end{aligned} \qquad \begin{pmatrix} x_{t+h} \\ v_{t+h} \end{pmatrix} = \begin{pmatrix} 1 & h \\ -hk & 1 \end{pmatrix} \begin{pmatrix} x_t \\ v_t \end{pmatrix}$$

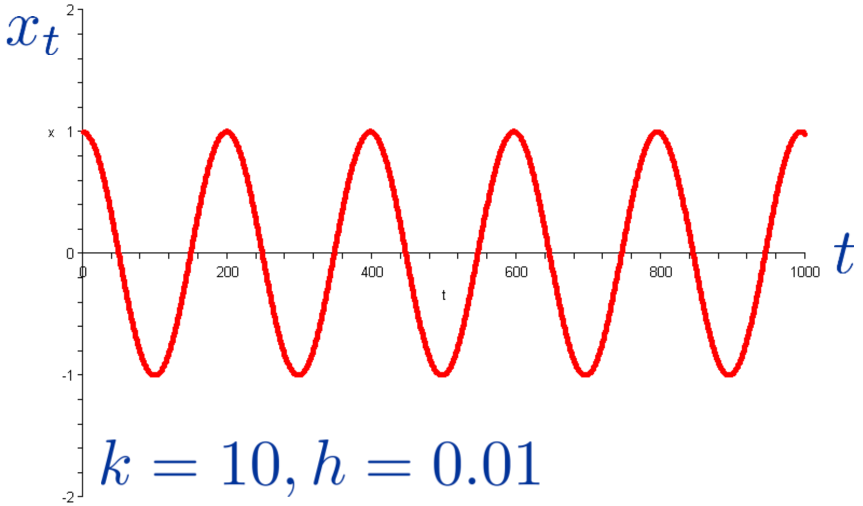
- solving $0 = \det(\mathbf{G} - \lambda \mathbf{I}) = \det \begin{pmatrix} 1-\lambda & h \\ -hk & 1-\lambda \end{pmatrix}$ to compute the eigenvalues
  $\lambda_{1,2} = 1 \pm h\sqrt{-k} = 1 \pm h\sqrt{k}i$
- the spectral radius is computed as
  $|\lambda_{1,2}| = \sqrt{\mathrm{Re}(\lambda_{1,2})^2 + \mathrm{Im}(\lambda_{1,2})^2} = \sqrt{1 + h^2 k} > 1$
- unconditionally unstable for undamped springs

Euler-Cromer / Explicit Euler

- Euler-Cromer

$$\mathbf{G} = \begin{pmatrix} 1 & h \\ -hk & 1 - h^2k \end{pmatrix}$$

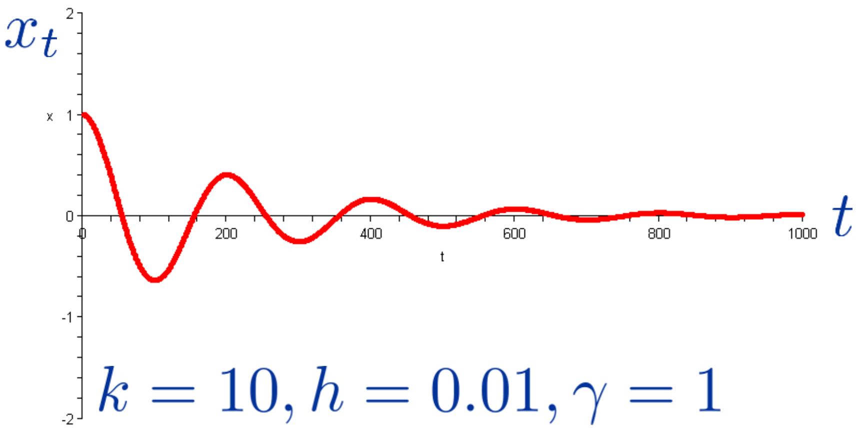
$$\rho = 1$$



- explicit Euler with damping

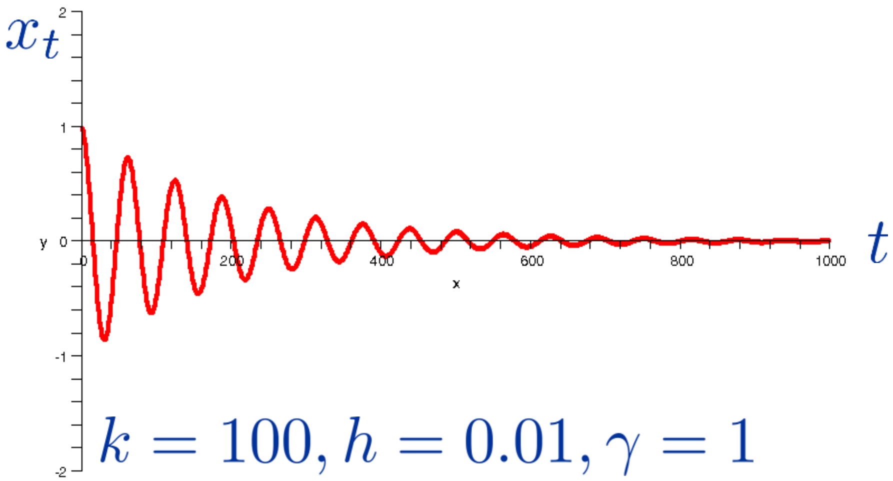
$$\mathbf{G} = \begin{pmatrix} 1 & h \\ -hk & 1 - h\gamma \end{pmatrix}$$

$$\rho \approx 0.9955$$

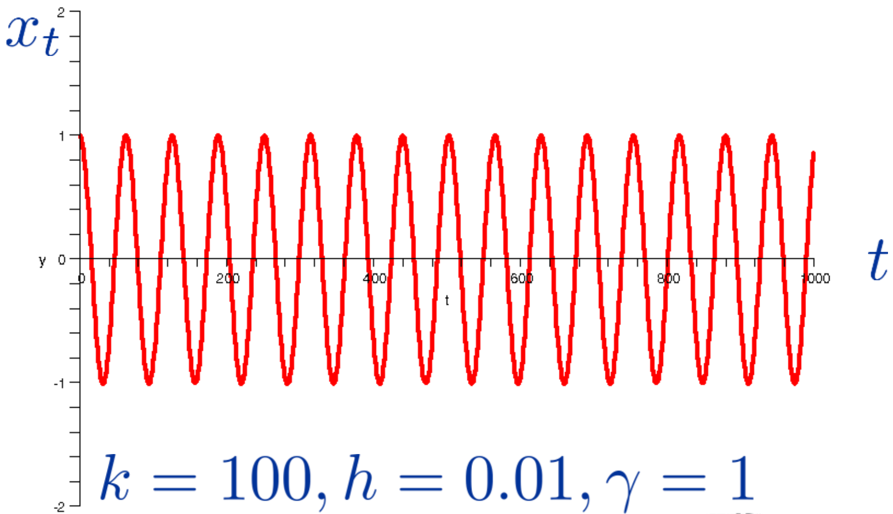


Euler-Cromer / Explicit Euler

- Euler-Cromer with damping
  $\rho \approx 0.9955$

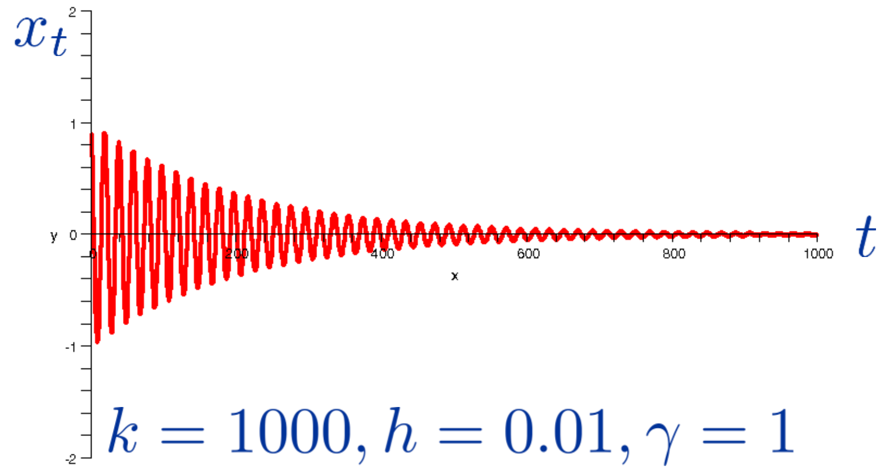


- explicit Euler with damping
  $\rho = 1$

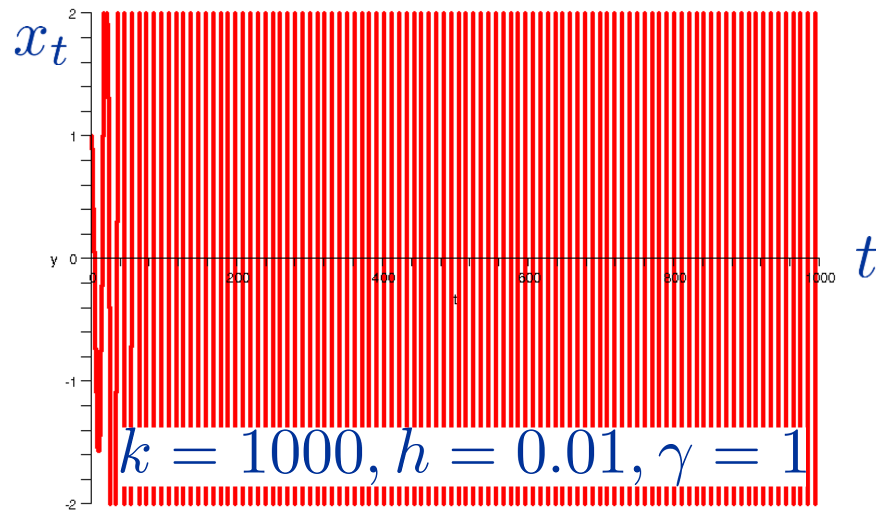


Euler-Cromer / Explicit Euler

- Euler-Cromer with damping
  $\rho \approx 0.9955$

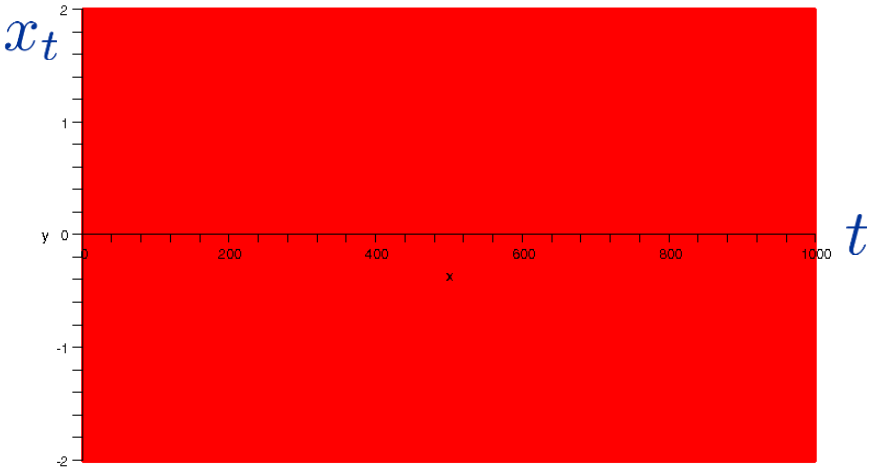


- explicit Euler with damping
  $\rho \approx 1.044$



Euler-Cromer

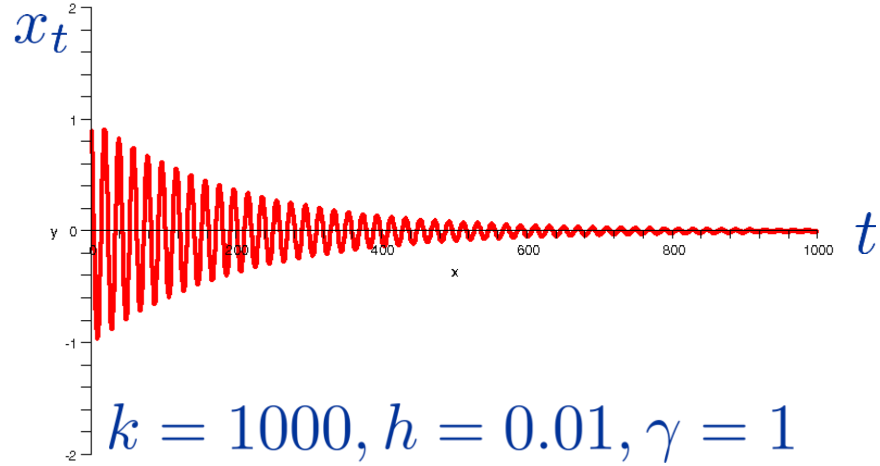
- Euler-Cromer with damping
  $\rho \approx 1.1465$



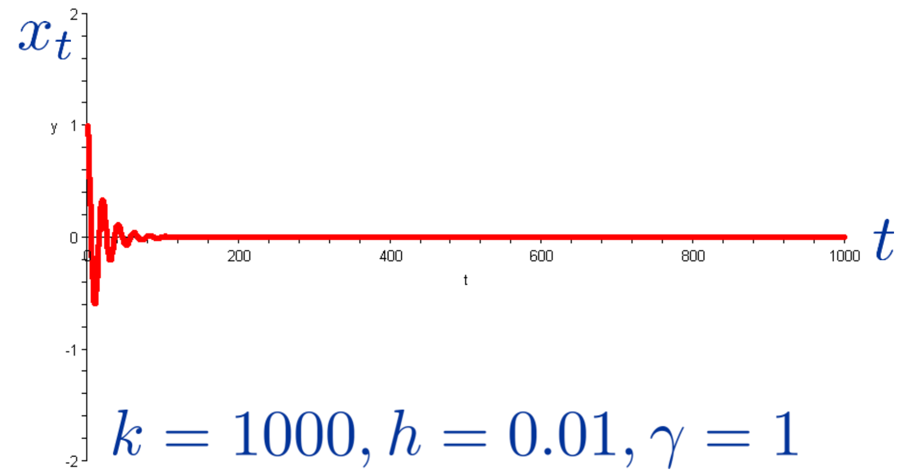
$k = 10^5, h = 0.01, \gamma = 1$

Euler-Cromer / Implicit Euler

- Euler-Cromer with damping
  $\rho \approx 0.9955$

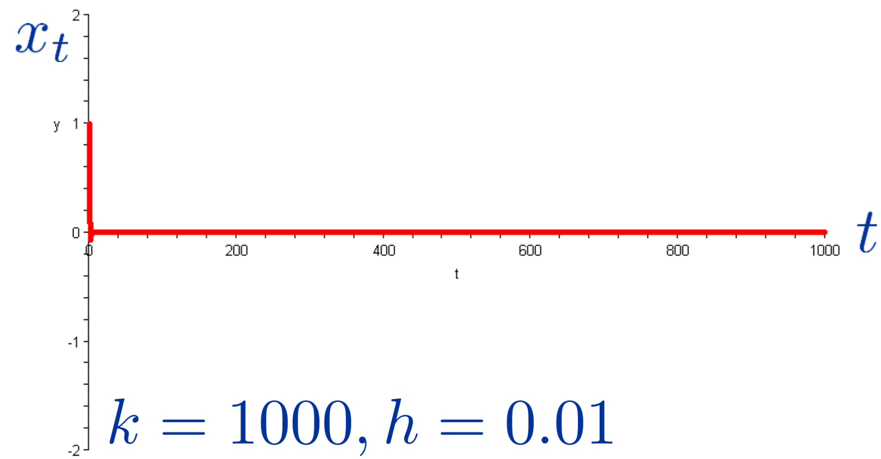


- implicit Euler with damping
  $\rho \approx 0.949$

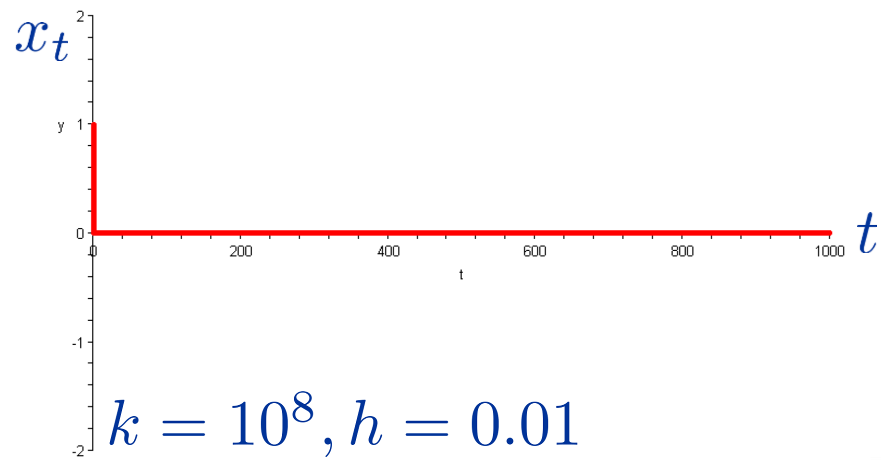


Implicit Euler without Damping

- implicit Euler without damping
  $\rho \approx 0.302$



- implicit Euler without damping
  $\rho \approx 0.01$



Implicit Euler without Damping

- amplification matrix

$$\mathbf{G} = \frac{1}{1+h^2k}\begin{pmatrix} 1 & h \\ -hk & 1 \end{pmatrix}$$

- eigenvalues

$$\lambda_{1,2} = \frac{1}{1+h^2k} \pm \frac{h\sqrt{k}}{1+h^2k} i$$

- spectral radius

$$\rho = \frac{1}{\sqrt{1+h^2k}} < 1$$

- unconditionally stable for undamped springs

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Discussion

- the eigenvalues depend on all parameters of a system, e. g. masses, deformation forces, damping forces, and the time step
- if a system is unconditionally stable / unstable, the stability is independent from the parameters
- if a system is conditionally stable, the spectral radius is smaller than one for certain parameter sets and one is interested in the maximum time step for a given parameter set
- for systems consisting of n mass points, $6n \times 6n$ matrices have to be analyzed

Discussion

- if an integration scheme is represented by a nonlinear transformation, the stability analysis is only approximate
- if any linearization is employed to derive the amplification
  matrix, the stability analysis is only approximate
- 3D mass-spring systems are non-linear

References

- Franz J. Vesely,
  "Computational Physics – An Introduction",
  Kluwer Academic, New York, ISBN 0-306-46631-7, 2001.
- Jonathan Richard Shewchuk,
  “An Introduction to the Conjugate Gradient Method Without the Agonizing Pain”, August 1994.

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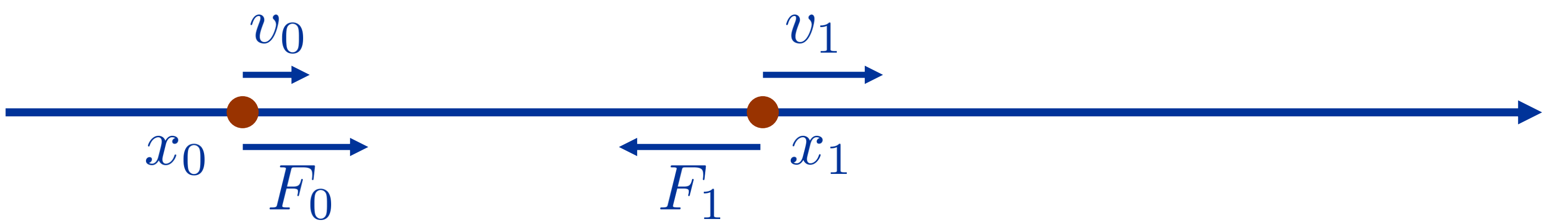
Example

- 1D mass points x_0, x_1 with masses m_0, m_1 connected with a spring of rest length L and stiffness k



- spring force $F_0^t = -F_1^t = k(x_1^t - x_0^t - L)$
- Euler-Cromer integration

$$
\begin{aligned}
x_0^{t+h} &= x_0^t + h v_0^{t+h} \\
x_1^{t+h} &= x_1^t + h v_1^{t+h}
\end{aligned}
\qquad
\begin{aligned}
v_0^{t+h} &= v_0^t + h \frac{F_0^t}{m_0} \\
v_1^{t+h} &= v_1^t - h \frac{F_0^t}{m_1}
\end{aligned}
$$

Example

- relative velocity $r^t = v_1^t - v_0^t$

$$r^{t+h} = v_1^t - h\frac{F_0^t}{m_1} - v_0^t - h\frac{F_0^t}{m_0}$$

$$r^{t+h} = r^t - hF_0^t\left(\frac{1}{m_0} + \frac{1}{m_1}\right) = r^t - hF_0^t m$$

- distance $d^t = x_1^t - x_0^t$

$$d^{t+h} = x_1^t + hv_1^{t+h} - x_0^t - hv_0^{t+h} = d^t + hr^{t+h}$$

$$d^{t+h} = d^t + h(r^t - hF_0^t m)$$

Example

$$F_0^t = k(x_1^t - x_0^t - L) = k(d^t - L)$$

- transformation of r^t and d^t

$$\begin{aligned} r^{t+h} &= r^t - hF_0^t m = r^t - hmk(d^t - L) = r^t - hmkd^t + hmkL \\ d^{t+h} &= d^t + h\big(r^t - hmk(d^t - L)\big) = hr^t + (1 - h^2mk)d^t + h^2mkL \end{aligned}$$

- amplification matrix

$$\begin{pmatrix} r^{t+h} \\ d^{t+h} \end{pmatrix} = \begin{pmatrix} 1 & -hmk \\ h & 1 - h^2mk \end{pmatrix} \begin{pmatrix} r^t \\ d^t \end{pmatrix} + \begin{pmatrix} hmkL \\ h^2mkL \end{pmatrix}$$

- conditionally stable