An example of an implicit Euler implementation

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1 Setting

Figure 1: Example setting with four particles and three springs.

Fig. 1 illustrates an example with four particles at positions $x_1, \ldots, x_4$ with masses $m_1, \ldots, m_4$ and three springs with rest lengths $L_1, \ldots, L_3$ and stiffness constants $k_1, k_2, k_3$. Gravity $g$, other accelerations $a_{\text{other}}$ that do not depend on positions, e.g., damping of relative velocities, and spring forces $F_{\text{spring}}^i$ are considered at particles. E.g., the acceleration $a_1^t$ at particle $x_1^t$ at time $t$ is computed as

$$a_1^t = g + a_{\text{other}}^t + \frac{1}{m_1} F_{\text{spring}}^{\text{spring},t}$$

with

$$F_{\text{spring}}^{\text{spring},t} = \frac{k_1}{L_1} \left( x_2^t - x_1^t - L_1 \frac{x_2^t - x_1^t}{|x_2^t - x_1^t|} \right) + \frac{k_3}{L_3} \left( x_3^t - x_1^t - L_3 \frac{x_3^t - x_1^t}{|x_3^t - x_1^t|} \right).$$

Note that $x_j^t - x_i^t - L_k \frac{x_j^t - x_i^t}{|x_j^t - x_i^t|} = (|x_j^t - x_i^t| - L_k) \frac{x_j^t - x_i^t}{|x_j^t - x_i^t|}$. 

1
Implicit Euler computes velocities $\mathbf{v}_i^{t+h}$ at the next timestep $t+h$ by solving a linear system. The computed velocities are used to update the particle positions with $\mathbf{x}_i^{t+h} = \mathbf{x}_i^t + h \mathbf{v}_i^{t+h}$. The implicit formulation for the velocity computation according to Euler would be

$$\mathbf{v}_i^{t+h} = \mathbf{v}_i^t + h (\mathbf{g} + \mathbf{a}_{i,\text{other},t+h} + \frac{1}{m_i} \mathbf{F}_{i,\text{spring},t+h}).$$

(3)

This is typically considered as too complex and the following formulation is solved instead:

$$\mathbf{v}_i^{t+h} = \mathbf{v}_i^t + h (\mathbf{g} + \mathbf{a}_{i,\text{other},t}) + \frac{1}{m_i} \mathbf{F}_{i,\text{spring},t+h}.$$  

(4)

The term $\mathbf{v}_i^* = \mathbf{v}_i^t + h (\mathbf{g} + \mathbf{a}_{i,\text{other},t})$ is explicitly computed and referred to as predicted velocity. Instead of Eq. 4, the following formulation is solved:

$$\mathbf{v}_i^{t+h} = \mathbf{v}_i^* + \frac{1}{m_i} \mathbf{F}_{i,\text{spring},t+h}.$$  

(5)

We now consider vectors that represent all particles, e.g.

$$\mathbf{x} = \begin{pmatrix} x_1^t \\ x_2^t \\ x_3^t \\ x_4^t \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} v_1^t \\ v_2^t \\ v_3^t \\ v_4^t \end{pmatrix}, \quad \mathbf{F}_{\text{spring},t} = \begin{pmatrix} F_{\text{spring},1,t} \\ F_{\text{spring},2,t} \\ F_{\text{spring},3,t} \\ F_{\text{spring},4,t} \end{pmatrix}$$

(6)

Eq. 5 can now be written for the entire particle set:

$$M \mathbf{v}^{t+h} = M \mathbf{v}^* + h \mathbf{F}_{\text{spring},t+h}$$

(7)

with

$$M = \begin{pmatrix} m_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & m_1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & m_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & m_2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & m_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

(8)

of size $12 \times 12$ for our example setting.
3 Linearization

The formulation

\[ Mv^{t+h} = Mv^* + hF_{\text{spring},t+h} \quad (9) \]

is replaced by the approximation

\[ Mv^{t+h} = Mv^* + hF_{\text{spring},t} + hJ^t(x^{t+h} - x^t) \quad (10) \]

with \( J^t \) being a Jacobi matrix that represents all partial derivatives of all spring forces with respect to all positions. This linearization of the forces and the fact that \( x^{t+h} - x^t = hv^{t+h} \) results in a linear system with unknown velocities \( v^{t+h} \):

\[ (M - h^2J^t)v^{t+h} = Mv^* + hF_{\text{spring},t} \quad (11) \]

We omit the time index for quantities at time \( t \) to improve readability. We also replace \( F_{\text{spring}} \) with \( F \).

\[ (M - h^2J)v^{t+h} = Mv^* + hF \quad (12) \]

4 The Jacobi matrix

The Jacobi matrix consists of submatrices \( J_{i,j} \) of size \( 3 \times 3 \). In our example setting, we have

\[ J = \begin{pmatrix} J_{1,1} & J_{1,2} & J_{1,3} & J_{1,4} \\ J_{2,1} & J_{2,2} & J_{2,3} & J_{2,4} \\ J_{3,1} & J_{3,2} & J_{3,3} & J_{3,4} \\ J_{4,1} & J_{4,2} & J_{4,3} & J_{4,4} \end{pmatrix} \quad (13) \]

Each submatrix \( J_{i,j} \) encodes the spatial derivative of the force \( F_i \) with respect to the position \( x_j \), i.e. the dependency of \( F_i \) from \( x_j \). \( J_{i,j} \) encodes how the force \( F_i \) changes due a small displacement of \( x_j \).

If a force \( F_i \) does not depend from a position \( x_j \), the respective derivative \( J_{i,j} \) is zero. In our example, the force \( F_4 = m_4g + m_4a^\text{other} \) does not depend on any of the positions \( x_{1..4} \). I.e., \( J_{4,1..4} = 0 \). Further, the forces \( F_{1..3} \) do not depend on position \( x_4 \). Thus, \( J_{1..3,4} = 0 \). The forces \( F_{1..3} \) depend on positions \( x_{1..3} \). So, \( J_{1..3,1..3} \) have to be computed. The Jacobian in our example has the
following structure:

\[
J = \begin{pmatrix}
J_{1,1} & J_{1,2} & J_{1,3} & 0 \\
J_{2,1} & J_{2,2} & J_{2,3} & 0 \\
J_{3,1} & J_{3,2} & J_{3,3} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\] (14)

4.1 Spring forces

The spring forces at the particles are

\[
F_1 = \frac{k_1}{L_1} \left( x_2 - x_1 - L_1 \frac{x_2 - x_1}{|x_2 - x_1|} \right) + \frac{k_3}{L_3} \left( x_3 - x_1 - L_3 \frac{x_3 - x_1}{|x_3 - x_1|} \right)
\]

\[
F_2 = -\frac{k_1}{L_1} \left( x_2 - x_1 - L_1 \frac{x_2 - x_1}{|x_2 - x_1|} \right) + \frac{k_2}{L_2} \left( x_3 - x_2 - L_2 \frac{x_3 - x_2}{|x_3 - x_2|} \right)
\]

\[
F_3 = -\frac{k_2}{L_2} \left( x_3 - x_2 - L_2 \frac{x_3 - x_2}{|x_3 - x_2|} \right) - \frac{k_3}{L_3} \left( x_3 - x_1 - L_3 \frac{x_3 - x_1}{|x_3 - x_1|} \right)
\] (15)

Each particle is connected to two springs. If we have a spring \( k \) that connects two particles \( i \) and \( j \), the force at particle \( i \) can be denoted as \( f_k(x_i \leftarrow x_j) \). E.g.,

\[
f_1(x_1 \leftarrow x_2) = \frac{k_1}{L_1} \left( x_2 - x_1 - L_1 \frac{x_2 - x_1}{|x_2 - x_1|} \right)
\]

\[
f_2(x_2 \leftarrow x_3) = \frac{k_2}{L_2} \left( x_3 - x_2 - L_2 \frac{x_3 - x_2}{|x_3 - x_2|} \right)
\]

\[
f_3(x_1 \leftarrow x_3) = \frac{k_3}{L_3} \left( x_3 - x_1 - L_3 \frac{x_3 - x_1}{|x_3 - x_1|} \right)
\] (16)

The spring forces are symmetric and sum up to zero, i.e.

\[
f_k(x_i \leftarrow x_j) = -f_k(x_j \leftarrow x_i).
\] (17)

The spring forces at the three particles can now be written as

\[
F_1 = f_1(x_1 \leftarrow x_2) + f_3(x_1 \leftarrow x_3)
\]

\[
F_2 = f_1(x_2 \leftarrow x_1) + f_2(x_2 \leftarrow x_3) = -f_1(x_1 \leftarrow x_2) + f_2(x_2 \leftarrow x_3)
\]

\[
F_3 = f_2(x_3 \leftarrow x_2) + f_3(x_3 \leftarrow x_1) = -f_2(x_2 \leftarrow x_3) - f_3(x_1 \leftarrow x_3).
\] (18)

Due to the symmetry, one force computation per spring is sufficient to assemble the overall forces at particles.
4.2 Spatial derivatives of spring forces

We consider force $F_1$ at particle $x_1$. We have to compute the spatial derivatives of $F_1 = (F_{1,x}, F_{1,y}, F_{1,z})^T$ with respect to positions $x_{1,3} = (x_{1,3,x}, x_{1,3,y}, x_{1,3,z})^T$. I.e.

$$J_{1,1} = \begin{pmatrix} \frac{\partial F_{1,x}}{\partial x_1} & \frac{\partial F_{1,y}}{\partial x_1} & \frac{\partial F_{1,z}}{\partial x_1} \\ \frac{\partial F_{1,x}}{\partial x_2} & \frac{\partial F_{1,y}}{\partial x_2} & \frac{\partial F_{1,z}}{\partial x_2} \\ \frac{\partial F_{1,x}}{\partial x_3} & \frac{\partial F_{1,y}}{\partial x_3} & \frac{\partial F_{1,z}}{\partial x_3} \end{pmatrix}, \quad J_{1,2} = \begin{pmatrix} \frac{\partial F_{1,x}}{\partial x_2} & \frac{\partial F_{1,y}}{\partial x_2} & \frac{\partial F_{1,z}}{\partial x_2} \\ \frac{\partial F_{1,x}}{\partial x_3} & \frac{\partial F_{1,y}}{\partial x_3} & \frac{\partial F_{1,z}}{\partial x_3} \end{pmatrix}, \quad J_{1,3} = \ldots$$

The matrices can also be rewritten using $f$ instead of $F$. E.g.,

$$J_{1,1} = \begin{pmatrix} \frac{\partial f_1(x_1 \leftarrow x_2)}{\partial x_1} & \frac{\partial f_3(x_1 \leftarrow x_3)}{\partial x_1} \\ \frac{\partial f_1(x_1 \leftarrow x_2)}{\partial x_2} & \frac{\partial f_3(x_1 \leftarrow x_3)}{\partial x_2} \\ \frac{\partial f_1(x_1 \leftarrow x_2)}{\partial x_3} & \frac{\partial f_3(x_1 \leftarrow x_3)}{\partial x_3} \end{pmatrix} + \begin{pmatrix} \frac{\partial f_3(x_1 \leftarrow x_3)}{\partial x_1} & \frac{\partial f_3(x_1 \leftarrow x_3)}{\partial x_2} \\ \frac{\partial f_3(x_1 \leftarrow x_3)}{\partial x_2} & \frac{\partial f_3(x_1 \leftarrow x_3)}{\partial x_3} \end{pmatrix}$$

(20)

and

$$J_{1,2} = \frac{\partial f_1(x_1 \leftarrow x_2)}{\partial x_2} + \frac{\partial f_3(x_1 \leftarrow x_3)}{\partial x_2}, \quad J_{1,3} = \frac{\partial f_1(x_1 \leftarrow x_2)}{\partial x_3} + \frac{\partial f_3(x_1 \leftarrow x_3)}{\partial x_3}. \quad (21)$$

We have already discussed that $J_{1,4} = 0$, because there is no spring between particles 1 and 4. This fact can also be seen from

$$J_{1,4} = \frac{\partial f_1(x_1 \leftarrow x_2)}{\partial x_4} + \frac{\partial f_3(x_1 \leftarrow x_3)}{\partial x_4} = 0 \quad (22)$$

as the forces $f_1(x_1 \leftarrow x_2)$ and $f_3(x_1 \leftarrow x_3)$ do not depend on $x_4$. The same argument can be used to see that $J_{1,2}$ and $J_{1,3}$ simplify to

$$J_{1,2} = \frac{\partial f_1(x_1 \leftarrow x_2)}{\partial x_2}, \quad J_{1,3} = \frac{\partial f_3(x_1 \leftarrow x_3)}{\partial x_3}. \quad (23)$$

Generally,

$$J_{1,j} = \sum_k \frac{\partial f_k(x_i \leftarrow x_m)}{\partial x_j}. \quad (24)$$

We sum over all springs $k$ that are connected to particle $i$. If a spring $k$ connects particles $i$ and $m$ and $j \neq i$ and $j \neq m$, then $\frac{\partial f_k(x_i \leftarrow x_m)}{\partial x_j} = 0$. 

5
4.3 Derivative of \( f_1(x_1 \leftarrow x_2) \) with respect to \( x_1 \)

The components of \( f_1(x_1 \leftarrow x_2) \) are

\[
f_{1,x} = \frac{k_1}{L_1} \left( x_{2,x} - x_{1,x} - L_1 \frac{x_{2,x} - x_{1,x}}{\sqrt{(x_{2,x} - x_{1,x})^2 + (x_{2,y} - x_{1,y})^2 + (x_{2,z} - x_{1,z})^2}} \right)
\]

\[
f_{1,y} = \frac{k_1}{L_1} \left( x_{2,y} - x_{1,y} - L_1 \frac{x_{2,y} - x_{1,y}}{\sqrt{(x_{2,x} - x_{1,x})^2 + (x_{2,y} - x_{1,y})^2 + (x_{2,z} - x_{1,z})^2}} \right)
\]

\[
f_{1,z} = \frac{k_1}{L_1} \left( x_{2,z} - x_{1,z} - L_1 \frac{x_{2,z} - x_{1,z}}{\sqrt{(x_{2,x} - x_{1,x})^2 + (x_{2,y} - x_{1,y})^2 + (x_{2,z} - x_{1,z})^2}} \right).
\]

(25)

We define

\[
l_{i,j} = \sqrt{(x_{j,x} - x_{i,x})^2 + (x_{j,y} - x_{i,y})^2 + (x_{j,z} - x_{i,z})^2}
\]

(26)

and the partial derivatives of \( f_1(x_1 \leftarrow x_2) \) with respect to \( x_1 \) can be written as

\[
\frac{\partial f_{1,x}}{\partial x_{1,x}} = \frac{k_1}{L_1} \left( -1 + \frac{L_1}{l_{1,2}} \left( 1 - \frac{(x_{2,x} - x_{1,x}) (x_{2,x} - x_{1,x})}{(l_{1,2})^2} \right) \right)
\]

\[
\frac{\partial f_{1,x}}{\partial x_{1,y}} = \frac{k_1}{L_1} \left( 0 + \frac{L_1}{l_{1,2}} \left( 0 - \frac{(x_{2,x} - x_{1,x}) (x_{2,y} - x_{1,y})}{(l_{1,2})^2} \right) \right)
\]

\[
\frac{\partial f_{1,x}}{\partial x_{1,z}} = \frac{k_1}{L_1} \left( 0 + \frac{L_1}{l_{1,2}} \left( 0 - \frac{(x_{2,x} - x_{1,x}) (x_{2,z} - x_{1,z})}{(l_{1,2})^2} \right) \right)
\]

\[
\frac{\partial f_{1,y}}{\partial x_{1,x}} = \frac{k_1}{L_1} \left( 0 + \frac{L_1}{l_{1,2}} \left( 0 - \frac{(x_{2,y} - x_{1,y}) (x_{2,x} - x_{1,x})}{(l_{1,2})^2} \right) \right)
\]

\[
\frac{\partial f_{1,y}}{\partial x_{1,y}} = \frac{k_1}{L_1} \left( 0 + \frac{L_1}{l_{1,2}} \left( 0 - \frac{(x_{2,y} - x_{1,y}) (x_{2,y} - x_{1,y})}{(l_{1,2})^2} \right) \right)
\]

\[
\frac{\partial f_{1,y}}{\partial x_{1,z}} = \frac{k_1}{L_1} \left( 0 + \frac{L_1}{l_{1,2}} \left( 0 - \frac{(x_{2,y} - x_{1,y}) (x_{2,z} - x_{1,z})}{(l_{1,2})^2} \right) \right)
\]

\[
\frac{\partial f_{1,z}}{\partial x_{1,x}} = \frac{k_1}{L_1} \left( 0 + \frac{L_1}{l_{1,2}} \left( 0 - \frac{(x_{2,z} - x_{1,z}) (x_{2,x} - x_{1,x})}{(l_{1,2})^2} \right) \right)
\]

\[
\frac{\partial f_{1,z}}{\partial x_{1,y}} = \frac{k_1}{L_1} \left( 0 + \frac{L_1}{l_{1,2}} \left( 0 - \frac{(x_{2,z} - x_{1,z}) (x_{2,y} - x_{1,y})}{(l_{1,2})^2} \right) \right)
\]

\[
\frac{\partial f_{1,z}}{\partial x_{1,z}} = \frac{k_1}{L_1} \left( 0 + \frac{L_1}{l_{1,2}} \left( 0 - \frac{(x_{2,z} - x_{1,z}) (x_{2,z} - x_{1,z})}{(l_{1,2})^2} \right) \right).
\]

(27)
Finally, we have

\[
\frac{\partial f_1(\mathbf{x}_1 \leftarrow \mathbf{x}_2)}{\partial \mathbf{x}_1} = \frac{\partial f_1}{\partial \mathbf{x}_{1,x}} \frac{\partial \mathbf{x}_{1,x}}{\partial \mathbf{x}_1} + \frac{\partial f_1}{\partial \mathbf{x}_{1,y}} \frac{\partial \mathbf{x}_{1,y}}{\partial \mathbf{x}_1} + \frac{\partial f_1}{\partial \mathbf{x}_{1,z}} \frac{\partial \mathbf{x}_{1,z}}{\partial \mathbf{x}_1}
\]

\[
= k_1 \left( -I + \frac{L_1}{\|\mathbf{x}_2 - \mathbf{x}_1\|} \left( I - \frac{(\mathbf{x}_2 - \mathbf{x}_1)(\mathbf{x}_2 - \mathbf{x}_1)^T}{\|\mathbf{x}_2 - \mathbf{x}_1\|^2} \right) \right)
\]  

(28)

4.4 Assembly of \( J \)

It is sufficient to compute one 3 \( \times \) 3 Jacobian per spring, i.e. 3 Jacobians in our example.

\[
\frac{\partial f_1(\mathbf{x}_1 \leftarrow \mathbf{x}_2)}{\partial \mathbf{x}_1} = \frac{k_1}{L_1} \left( -I + \frac{L_1}{\|\mathbf{x}_2 - \mathbf{x}_1\|} \left( I - \frac{(\mathbf{x}_2 - \mathbf{x}_1)(\mathbf{x}_2 - \mathbf{x}_1)^T}{\|\mathbf{x}_2 - \mathbf{x}_1\|^2} \right) \right)
\]

\[
\frac{\partial f_2(\mathbf{x}_2 \leftarrow \mathbf{x}_3)}{\partial \mathbf{x}_2} = \frac{k_2}{L_2} \left( -I + \frac{L_2}{\|\mathbf{x}_3 - \mathbf{x}_2\|} \left( I - \frac{(\mathbf{x}_3 - \mathbf{x}_2)(\mathbf{x}_3 - \mathbf{x}_2)^T}{\|\mathbf{x}_3 - \mathbf{x}_2\|^2} \right) \right)
\]

\[
\frac{\partial f_3(\mathbf{x}_1 \leftarrow \mathbf{x}_3)}{\partial \mathbf{x}_1} = \frac{k_3}{L_3} \left( -I + \frac{L_3}{\|\mathbf{x}_3 - \mathbf{x}_1\|} \left( I - \frac{(\mathbf{x}_3 - \mathbf{x}_1)(\mathbf{x}_3 - \mathbf{x}_1)^T}{\|\mathbf{x}_3 - \mathbf{x}_1\|^2} \right) \right)
\]  

(29)

All submatrices \( J_{i,j} \) can be assembled from these three Jacobians. If a spring \( k \) connects particles \( i \) and \( j \), we have

\[
\frac{\partial f_k(\mathbf{x}_i \leftarrow \mathbf{x}_j)}{\partial \mathbf{x}_i} = \frac{\partial f_k}{\partial \mathbf{x}_{i,x}} \frac{\partial \mathbf{x}_{i,x}}{\partial \mathbf{x}_i} + \frac{\partial f_k}{\partial \mathbf{x}_{i,y}} \frac{\partial \mathbf{x}_{i,y}}{\partial \mathbf{x}_i} + \frac{\partial f_k}{\partial \mathbf{x}_{i,z}} \frac{\partial \mathbf{x}_{i,z}}{\partial \mathbf{x}_i}
\]

\[
= -\frac{\partial f_k}{\partial \mathbf{x}_{j,x}} \frac{\partial \mathbf{x}_{j,x}}{\partial \mathbf{x}_j} - \frac{\partial f_k}{\partial \mathbf{x}_{j,y}} \frac{\partial \mathbf{x}_{j,y}}{\partial \mathbf{x}_j} - \frac{\partial f_k}{\partial \mathbf{x}_{j,z}} \frac{\partial \mathbf{x}_{j,z}}{\partial \mathbf{x}_j}
\]

(30)

Again, each submatrix \( J_{i,j} \) is computed by summing up contributions from all springs \( k \) that are connected to particle \( i \):

\[
J_{i,j} = \sum_k \frac{\partial f_k(\mathbf{x}_i \leftarrow \mathbf{x}_m)}{\partial \mathbf{x}_j}.
\]  

(31)

We sum over all springs \( k \) that are connected to particle \( i \). If \( i = j \), we have

\[
J_{i,i} = \sum_k \frac{\partial f_k(\mathbf{x}_i \leftarrow \mathbf{x}_m)}{\partial \mathbf{x}_i}.
\]  

(32)

In this case, the sum contains as many elements as there are springs connected to particle \( i \). If no spring is connected to particle \( i \), e.g., particle 4 in our example, we have \( J_{i,i} = 0 \). If \( i \neq j \), we have \( J_{i,j} = 0 \), if there is no spring between \( i \) and \( j \). If there is one spring, we have

\[
J_{i,j} = \frac{\partial f_k(\mathbf{x}_i \leftarrow \mathbf{x}_j)}{\partial \mathbf{x}_j}.
\]  

(33)
4.5 The final $J$

We define

\[ S_1 = \frac{\partial f_1(x_1 \leftarrow x_2)}{\partial x_1} \]
\[ S_2 = \frac{\partial f_2(x_2 \leftarrow x_3)}{\partial x_2} \]
\[ S_3 = \frac{\partial f_3(x_1 \leftarrow x_3)}{\partial x_1} \]

and we remember

\[ S_k = \frac{\partial f_k(x_i \leftarrow x_j)}{\partial x_i} = \frac{\partial f_k(x_j \leftarrow x_i)}{\partial x_j} = -\frac{\partial f_k(x_i \leftarrow x_j)}{\partial x_i} = -\frac{\partial f_k(x_j \leftarrow x_i)}{\partial x_j} \]

The final form of our exemplary Jacobian $J$ is now

\[ J = \begin{pmatrix} S_1 + S_3 & -S_1 & -S_3 & 0 \\ -S_1 & S_1 + S_2 & -S_2 & 0 \\ -S_3 & -S_2 & S_2 + S_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \] (36)

Each spring influences forces at two particles and depends on two particle positions. Thus, $S_i$ occurs four times in $J$ for a spring $i$.

4.6 The final system

Our system

\[ (M - h^2 J) v^{t+h} = M v^* + h F \]

(37)

represents four particles and can be written as $A v^{t+h} = s$ or

\[ \begin{pmatrix} M_1 - h^2 J_{1,1} & -h^2 J_{1,2} & -h^2 J_{1,3} & -h^2 J_{1,4} \\ -h^2 J_{2,1} & M_2 - h^2 J_{2,2} & -h^2 J_{2,3} & -h^2 J_{2,4} \\ -h^2 J_{3,1} & -h^2 J_{3,2} & M_3 - h^2 J_{3,3} & -h^2 J_{3,4} \\ -h^2 J_{4,1} & -h^2 J_{4,2} & -h^2 J_{4,3} & M_4 - h^2 J_{4,4} \end{pmatrix} \begin{pmatrix} v_1^{t+h} \\ v_2^{t+h} \\ v_3^{t+h} \\ v_4^{t+h} \end{pmatrix} = \begin{pmatrix} h F_1 \\ h F_2 \\ h F_3 \\ h F_4 \end{pmatrix} \] (38)
with

\[ M_i = \begin{pmatrix} m_i & 0 & 0 \\ 0 & m_i & 0 \\ 0 & 0 & m_i \end{pmatrix} \] \hspace{1cm} (39)

or

\[
\begin{pmatrix}
A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\
A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\
A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\
A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4}
\end{pmatrix}
\begin{pmatrix}
v_{i+1}^1 \\
v_{i+1}^2 \\
v_{i+1}^3 \\
v_{i+1}^4
\end{pmatrix} =
\begin{pmatrix}
s_1 \\
s_2 \\
s_3 \\
s_4
\end{pmatrix}
\] \hspace{1cm} (40)

### 4.7 Jacobi solver

The Jacobi update is

\[ v^{l+1} = v^l + \omega D^{-1}(s - Av^l) \] \hspace{1cm} (41)

with \( l \) being the iteration count, \( 0 < \omega \leq 0.5 \) being a coefficient and \( D \) being a matrix with the diagonal elements of \( A \). This update can be written as

\[
\begin{pmatrix}
v_{i+1}^1 \\
v_{i+1}^2 \\
v_{i+1}^3 \\
v_{i+1}^4
\end{pmatrix} =
\begin{pmatrix}
v_1^l \\
v_2^l \\
v_3^l \\
v_4^l
\end{pmatrix} + 
\omega
\begin{pmatrix}
D_1 & 0 & 0 & 0 \\
0 & D_2 & 0 & 0 \\
0 & 0 & D_3 & 0 \\
0 & 0 & 0 & D_4
\end{pmatrix}^{-1}
\begin{pmatrix}
s_1 \\
s_2 \\
s_3 \\
s_4
\end{pmatrix} - 
\begin{pmatrix}
A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\
A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\
A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\
A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4}
\end{pmatrix}
\begin{pmatrix}
v_1^l \\
v_2^l \\
v_3^l \\
v_4^l
\end{pmatrix} \] \hspace{1cm} (42)

with \( D_i \) representing the diagonal elements of \( A_{i,i} \). The update could be implemented on a per particle basis, i.e.:

\[ v_i^{l+1} = v_i^l + \omega D_i^{-1} \left( s_i - \sum_{j=1..4} A_{i,j} v_j^l \right) \] \hspace{1cm} (43)

This requires to represent \( A_{i,j} \). As \( A \) is sparsely filled in larger practical scenarios, \( A_{i,j} = 0 \) for many \( i \) and \( j \). Alternatively,

\[
\begin{pmatrix}
A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\
A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\
A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\
A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4}
\end{pmatrix}
\begin{pmatrix}
v_1^l \\
v_2^l \\
v_3^l \\
v_4^l
\end{pmatrix} =
\begin{pmatrix}
t_1 \\
t_2 \\
t_3 \\
t_4
\end{pmatrix}
\] \hspace{1cm} (44)
could be implemented by collecting the contributions from particles and springs. In our example,

\[
A = \begin{pmatrix}
M_1 & 0 & 0 & 0 \\
0 & M_2 & 0 & 0 \\
0 & 0 & M_3 & 0 \\
0 & 0 & 0 & M_4
\end{pmatrix}
- h^2
\begin{pmatrix}
(S_1 + S_3) & -S_1 & -S_3 & 0 \\
-S_1 & (S_1 + S_2) & -S_2 & 0 \\
-S_3 & -S_2 & (S_2 + S_3) & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\] (45)

\[S_k = \frac{\partial f_k(x_i \leftarrow x_j)}{\partial x_i}\] with \(i\) and \(j\) being the two particles of spring \(k\).

Initialization: Set \(t_i = 0\) for all particles \(i\). For all springs \(k\), compute \(S_k = \frac{\partial f_k(x_i \leftarrow x_j)}{\partial x_i}\)

Iterate over particles \(i\): Compute \(t_i+ = M_i v_i^l = m_i v_i^l\).

Iterate over springs \(k\): Spring \(k\) contributes to forces \(i\) and \(j\) and connects particles \(i\) and \(j\) corresponding to four contributions in matrix \(J\):

\[S_k = \frac{\partial f_k(x_i \leftarrow x_j)}{\partial x_i} = \frac{\partial f_k(x_j \leftarrow x_i)}{\partial x_j} = -\frac{\partial f_k(x_i \leftarrow x_j)}{\partial x_i} = -\frac{\partial f_k(x_j \leftarrow x_i)}{\partial x_j}\]

\[
Compute \begin{align*}
t_i^+ &= -h^2 \left( \frac{\partial f_k(x_i \leftarrow x_j)}{\partial x_i} v_i^l + \frac{\partial f_k(x_j \leftarrow x_i)}{\partial x_j} v_j^l \right) \\
&= -h^2 (S_k v_i^l - S_k v_j^l) = -h^2 S_k (v_i^l - v_j^l) \\
t_j^+ &= -h^2 \left( \frac{\partial f_k(x_j \leftarrow x_i)}{\partial x_i} v_i^l + \frac{\partial f_k(x_i \leftarrow x_j)}{\partial x_j} v_j^l \right) \\
&= -h^2 (-S_k v_i^l + S_k v_j^l) = -h^2 S_k (v_j^l - v_i^l) 
\end{align*}\]

Iterate over particles \(i\): Compute

\[
v_i^{l+1} = v_i^l + \omega D_i^{-1} (s_i - t_i)\]

Note: Matrix \(D_i\) contains the three diagonal elements \(a_{11}, a_{22}, a_{33}\), of \(A_{i,j}\).

I.e.,

\[
D_i^{-1} = \begin{pmatrix}
\frac{1}{a_{11}} & 0 & 0 \\
0 & \frac{1}{a_{22}} & 0 \\
0 & 0 & \frac{1}{a_{33}}
\end{pmatrix}
\] (49)

These values can be precomputed and stored at particles.